Codimension one foliations related to contact topology in low-dimensional manifolds via Open Books

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled " Codimension one foliations related to contact topology in low-dimensional manifolds via Open Books " towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Sayantika Mondal at The Australian National University under the supervision of Joan Licata, Senior Lecturer, MSI, during the academic year 2019-2020.

Joan Licata

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This thesis is dedicated to the people who made Canberra home.

Declaration

I hereby declare that the matter embodied in the report entitled "Codimension one foliations related to contact topology in low-dimensional manifolds via Open Books " are the results of the work carried out by me at the Mathematical Sciences Institute, The Australian National University, under the supervision of Joan Licata and the same has not been submitted elsewhere for any other degree.

Sayantika Mondal

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Sayantika

Abstract

In this study we have been trying to understand various aspects of Contact Geometry in a contact-3 manifold setting. We began by looking at the differential topology aspects of contact manifolds and their relation to more topological objects like Knots and Braids in contact 3-manifolds, relation between foliations and contact structures. A contact structure is a non-integrable plane field on a 3-manifold. An "Open Book" is an important tool that serves as a bridge between the differential geometric side of contact geometry and the cut-and-paste methods of low-dimensional topology. An "Open Book" is a topological decomposition of a 3-manifold that also specifies an equivalence class of contact structures on the manifold. Furthermore, when contact structures are viewed only as a homotopy classes of plane fields, we can consider foliations in the same class and explore their relations. We explore in details relation between contact structures and their relation to codimension 1 foliations, in particular the construction of a foliation close to any given contact structure. We study other related foliations and conclude whether it perturbs to a tight or overtwisted contact structure.

Contents

Introduction

The field of contact topology started with Huygens, Hamilton and Jacobi's work on geometric optics. Over the course of the years it has been studied in great detail by mathematicians such as Lie, Cartan and Darboux. Topological aspects of the subject have been more widely studied in the last few decades. In this thesis we have broadly been trying to understand various aspects of contact geometry in a contact-3 manifold setting. A contact structure is a non-integrable plane field on a 3-manifold. We begin discussing differential geometric aspects of contact structures in details with an eye towards exploring their relations with more topological objects like knots and braids in contact 3-manifolds and relation and foliations. In Chaper 1, we outline basic definitions,theorems, examples and tools to understand and classify contact structures. The results we discuss can be found in [8], [13].

An "Open Book" is a topological decomposition of a 3-manifold that also specifies an equivalence class of contact structures on the manifold. This serves as a key tool in our study by forming a bridge between the differential geometric side of contact geometry and the cut-and-paste methods of low-dimensional topology. In Chapter 3 we discuss different aspects of Open Books following [10]

Furthermore, when contact structures are viewed only as a homotopy classes of plane fields, we can consider foliations in the same class and explore their relations. We explore connections between plane fields in greater generality in Chapter 4. We discuss various relations between contact structures, foliations and confoliations. Our discussion on confoliations is based on [17]

In Chapter 5, we discuss relationship between codimension-1 foliations and contact structures following Etnyre's proof that every contact structure has a foliation close to it [11]. The proof relies on constructing an explicit foliation using a compatible Open Book for a given contact structure. Adapting ideas of the proof, we consider various related foliations and prove that under changing certain constraints we perturb to an overtwisted contact structure.

Chapter 1

Preliminaries

In this chapter we highlight basic ideas of differential geometry from the books "Calculus on Manifolds" by Michael Spivak and "An introduction to differentiable manifolds and Riemannian geometry" by William Boothby, which will be useful in understanding the rest of the thesis.

1.1 Manifolds

Definition 1.1.1. A n-manifold is a a second-countable Hausdorff topological space such that every point in the space has a neighbourhood that is homeomorphic to Euclidean n-space.

Definition 1.1.2. (Smooth Manifold) A differentiable or smooth manifold of dimension n is a topological manifold M, with a family $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$ of charts such that

1. $M = \cup_\alpha U_\alpha$
2. $(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta)$ are smoothly compatible.
3. If (V, ψ) is smoothly compatible with $(U_\alpha, \phi_\alpha)\, \forall \alpha$ implies $(V, \psi) \in \mathcal{U}$

Definition 1.1.3. (Smooth Functions) Let M be a smooth manifold and $U \subset M$. A function $f : U \to R$ is said to be smooth if there exists a chart (V, ψ) around p, $V \subset U$, such that

$f \circ \psi : \psi(V) \to R$ is a smooth function.

Definition 1.1.4. (Diffeomorphism) A diffeomorphism $f : M \to N$ is an isomorphism in the category of smooth manifolds, i.e., f is a smooth, and there exist a function $g : N \to M$ smooth such that $f \circ g = 1_N$, $g \circ f = 1_M$.

1.2 Tangent Space

Roughly speaking the tangent space of a manifold at a point is the collection of all tangent vectors at that point. This forms a vector space.

Definition 1.2.1. (Tangent Space) $T_p(M) =$ set of all point derivations of $\mathcal{C}_p^\infty(M)$.

Theorem 1.2.2. *$T_p(R^n)$ is isomorphic to R^n*

Corollary 1.2.3. *If $U \subset R^n$ then $T_p(U) \simeq R^n$*

1.2.1 Tangent Level Map / Differential

Let $F : M \to N$ be a smooth map of smooth manifolds. Take $p \in M$. Then F pulls back smooth functions on N to smooth functions on M and pushes forward tangent vectors.

The tangent level map (map between tangent spaces of M and N induced by F) of F is given by the Jacobian of F.

Definition 1.2.4. (Vector Field) A smooth vector field X on M is a function that assigns to each $p \in M$ a tangent vector at p in a smooth manner, i.e., the components of the tangent vector in the frame given by a local chart as p varies over the chart are smooth functions.

1.3 Exterior Algebra

Let V be a vector space over R, then the k-fold product $V \times \cdots \times V$ is denoted by V^k. A function $T : V^k \to R$ is called multilinear if it is linear in each component. This is also called a k-tensor on v. The set of all k-tensors is denoted by $\mathcal{T}^k(V)$.

Definition 1.3.1. (Tensor Product)If $M \in \mathcal{T}^k(V)$ and $N \in \mathcal{T}^l(V)$, we can define the tensor product, $M \otimes N \in \mathcal{T}^{k+l}(V)$ by,

$$M \otimes N(v_1, \cdots, v_k, v_{k+1}, \cdots, v_{k+l}) = M(v_1, \cdots, v_k) \cdot N(v_{k+1}, \cdots, v_{k+l})$$

.

Definition 1.3.2. (Alternating Tensor) A k-tensor ω is called alternating if

$$\omega(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = -\omega(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k)$$

The set of all alternating k-tensors is clearly a subspace denoted by $\Lambda_k(v)$.

Constructing alternating tensors

Consider, $M \in \mathcal{T}^k(V)$, we define $\mathrm{Alt}(M)$ as follows,

$$\mathrm{Alt}(M)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \cdot M\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right).$$

Theorem 1.3.3. *(Properties) If $M \in \mathcal{T}^k(V)$, we have the following,*

- $\mathrm{Alt}(M) \in \Lambda^k(V)$
- $\mathrm{Alt}(\mathrm{Alt}(M)) = \mathrm{Alt}(M)$
- *If $M \in \Lambda^k(V)$ then* $\mathrm{Alt}(M) = M$

Definition 1.3.4. A differential k-form is a function ω such that $\omega(p) \in \Lambda^k(R^n_p)$.

A differential form on a smooth manifold is defined similarly using local charts. The set of all differential k-forms on a manifold form vector space.

Alternatively, we can define a differential form as follows,

Definition 1.3.5. A smooth differential form of degree k, on a smooth manifold M is a smooth section of the k^{th} exterior power of its cotangent bundle. The set of all differential k-forms on a manifold form vector space.

If we have local coordinates $x_i, \ldots, x_n$ around a point p, then $dx^1(p), \ldots, dx^n(p)$ forms a dual basis to the standard basis of R^n_p. Then we can express any k-form ω at p as,

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

.

Differential operator

The differential operator d can be generalized to forms as an operator that transforms k-forms to $k+1$-forms. If ω is expressed as above, then dw is given by,

$$\begin{aligned} d\omega &= \sum_{i_1 < \cdots < i_{i_2}} d\omega_{i_1,\ldots,i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^{n} D_\alpha \left(\omega_{i_1,\ldots,i_k}\right) \cdot dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \end{aligned}$$

Theorem 1.3.6. *(Properties of d)*

- $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$
- *If ω_1 is a k-form and ω_2 is a l-form, then*

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$$

- $d(d(\omega)) = 0$

Chapter 2

Contact Structures

In this chapter we introduce our main object of interest - contact structures. We begin by looking at definitions and examples, local and global structures and types of contact structures. We also discuss how foliations, knots and surfaces in the manifold can be used to probe these structures. The characteristic foliation serves as an important tool in classifying contact structures by looking at a surface in the manifold, while knots and their invariants provide another interesting way of understanding and classifying contact structures. The material summarized here may be found , for example, in "Introductory Lectures in Contact Topology" and "Legendrian and transversal knots" by J.B. Etnyre.

2.1 Definitions and examples

Contact structures are plane fields on 3-dimensional manifolds satisfying certain conditions. More generally they are $(n-1)$ dimensional structures on odd dimensional n-manifolds.

Definition 2.1.1. A plane field ξ on a manifold M is a 2-dimensional subbundle of the tangent bundle TM.

Definition 2.1.2. A plane field ξ on a 3-manifold is called a contact structure if there exists a 1-form α with $\xi = \ker\alpha$ such that

$$\alpha \wedge d\alpha \neq 0$$

Alternatively, a contact structure ξ on a 3-manifold can be defined as a nowhere integrable plane field. The equivalence with the above definition can be established via Frobenius integrability.

Note: α can be defined locally or globally.

Definition 2.1.3. A contact structure is said to be positive if $\alpha \wedge d\alpha > 0$ and negative if $\alpha \wedge d\alpha < 0$.

For most purposes it is sufficient to just consider positive contact structures, and unless specified, we assume that our contact structure is positive. However, in particular situations we need to make a distinction between the two, as in Chapter 5.

Example 2.1.4. Standard contact structure on $\mathbb{R}^3$.

Consider $\mathbb{R}^3$ with standard Cartesian coordinates (x, y, z) and the 1-form $\alpha_1 = dz - ydx$. It can be easily checked that α_1 is a contact form and $\xi_1 = \ker\alpha_1$ is spanned by $\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial z} - \frac{\partial}{\partial y}\}$ at the point (x, y, z).

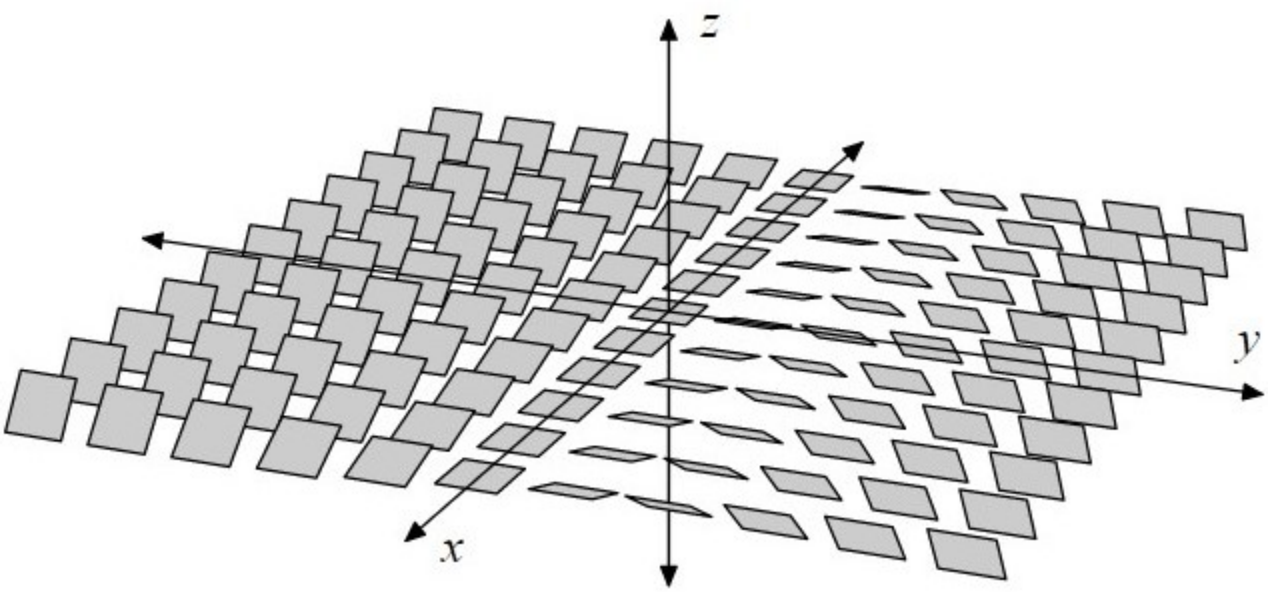


Figure 2.1: Standard contact structure on $\mathbb{R}^3$ [1]

[1]Public Domain, https://en.wikipedia.org/w/index.php?curid=21556952

Example 2.1.5. Radially symmetric contact structure.

Consider $\mathbb{R}^3$ with cylindrical coordinates (r, θ, z) and the 1-form $\alpha_2 = dz + r^2 d\theta$.

$\xi_2 = \ker \alpha_2$ is spanned by $\{\frac{\partial}{\partial r}, r^2 \frac{\partial}{\partial z} - \frac{\partial}{\partial \theta}\}$ at the point (r, θ, z). This structure is symmetric about the z -axis.

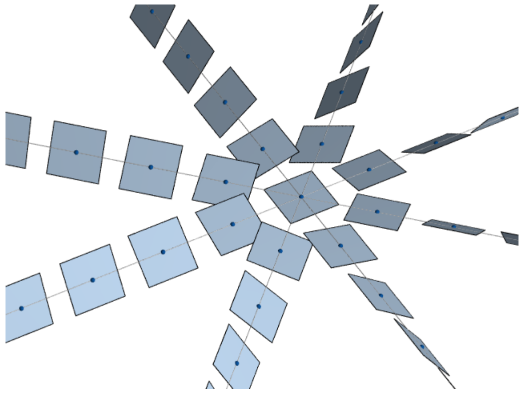


Figure 2.2: Radially symmetric contact structure on $\mathbb{R}^3$ [2]

Example 2.1.6. Overtwisted contact structure

Consider $\mathbb{R}^3$ with cylindrical coordinates (r, θ, z) and the 1-form $\alpha_3 = \cos r dz + r \sin r d\theta$.

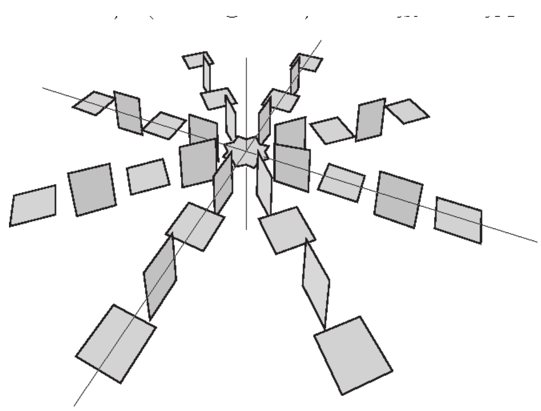


Figure 2.3: Overtwisted contact structure on $\mathbb{R}^3$ [3]

Definition 2.1.7. Two contact structures (M_1, ξ_1) and (M_2, ξ_2) are said to be contactomorphic if there is a diffeomorphism $f : M_1 \to M_2$ such that $f_*(\xi_1) = \xi_2$.

[2] Figure courtesy of P. Massot

[3] Figure courtesy of S. Schönenberger

Note: Alternatively in terms of the 1-form defining contact structures, contactomorphism is given by $f^*\alpha_2 = g\alpha_1$ for some positive smooth function $g : M_1 \to \mathbb{R}^{>0}$.

Examples 2.3 and 2.4 are contactomorphic while 2.5 is not contactomorphic to the rest.

Definition 2.1.8. (Reeb vector field) The Reeb vector field of a contact form α, v_α, is defined as the unique vector field v such that $\alpha(v) = 1$ and $d\alpha(v, \cdot) = 0$

The dynamical properties of the flow v_α are not preserved by contactomorphism.

2.2 Local Structure

Darboux's Theorem

This theorem essentially says that all contact structures look the same locally i.e. in an open neighbourhood of a point.

Theorem 2.2.1. *[1] Let (M, ξ) be any contact 3-manifold and p any point in (M, ξ). Then there exists neighbourhoods N of p in M, and U of $(0,0,0)$ in $(\mathbb{R}^3, dx - ydx)$ and a contactomorphism,*

$$f : (N, \xi|_N) \to (U, \xi_1|_U).$$

The comment after the examples claimed that not all contact structures are contactomorphic but Darboux's theorem tells us that even non-contactomorphic contact structures are same locally; thus we cannot have local invariants. As we will see next, the Gray's stability theorem says that there are no non-trivial deformations of contact structures on closed manifolds. However, this local flexibilty helps prove strong global results.

Gray's Theorem

Theorem 2.2.2. *(Theorem 2.20,[13]) Let $\{\xi_t\}_{t\in[0,1]}$ be a family of contact structures on a manifold M that differ on a compact set $C \subset int(M)$. Then there exists an isotopy $\psi_t : M \to M$ such that* *(i)* $(\psi_t)_* \xi_1 = \xi_t$ *(ii)* ψ_t *is the identity outside of an open neighborhood of C.*

Proof. The proof of this uses what is known as the Moser trick (Section 1.4, [5]). We assume ψ_t to be the flow of a vector field X_t. The equation for ψ_t can then be translated into an equation for X_t. If we can solve it, we can find ψ_t by integrating X_t.

If $\xi_t = \ker \alpha_t$, then ψ_t satisfies

$$\psi_t^* \alpha_t = \lambda_t \alpha_0$$

for some non-vanishing function $\lambda_t : M \to R$. Differentiating both side with respect to t and rearranging the terms we get,

$$\psi_t^* \left(\frac{d\alpha_t}{dt} + \mathcal{L}_{X_t} \alpha_t \right) = \frac{d\lambda_t}{dt} \alpha_0 = \frac{d\lambda_t}{dt} \frac{1}{\lambda_t} \psi^* \alpha_t$$

Using Cartan's formula this is equivalent to,

$$\psi_t^* \left(\frac{d\alpha_t}{dt} + d\left(\iota_{X_t} \alpha_t\right) + \iota_{X_t} d\alpha_t \right) = \psi_t^* \left(f_t \alpha_t\right) \text{ for } f_t = \frac{d}{dt} \left(\log \lambda_t\right) \circ \psi_t^{-1}$$

If X_t is chosen in ξ_t then $\iota X_t \alpha_t = 0$ and the above equation becomes,

$$\frac{d\alpha_t}{dt} + \iota_{X_t} d\alpha_t = f_t \alpha_t$$

Applying this to the Reeb vector field of α_t, v_{α_t} (that is, the unique vector field v_t such that $\alpha_t(v_t) = 1$ and $d\alpha_t(v_t, \cdot) = 0$), we find $f_t = \frac{d\alpha_t}{dt}(v_{\alpha_t})$ and X_t given by

$$\iota_{X_t} d\alpha_t = f_t \alpha_t - \frac{d\alpha_t}{dt}$$

The form $d\alpha_t$ gives an isomorphism

$$\begin{gathered} \Delta\left(\xi_t\right) \to \Omega_{\alpha_t} \\ v \mapsto \iota_v d\alpha_t \end{gathered}$$

where $\Delta(\xi_t) = \{v | v \in \xi_t\}$ and $\Omega_{\alpha_t} = \{ \text{ 1-forms } \beta | \beta(v_t) = 0\}$, and thus X_t is uniquely determined by the above equation. By construction, the flow of X_t gives us the required ψ_t.

For the subset of M where the ξ_t's agree we choose the α_t's to agree. This implies $\frac{d\alpha_t}{dt} = 0$, $f_t = 0$ and $X_t = 0$ and all equalities hold. $\square$

2.3 Characteristic Foliations

To understand the geometry of plane fields better, it's useful to look at the traces it leaves when intersected with a surface.

Let Σ be an embedded oriented surface in a contact manifold (M, ξ). At each point x of Σ consider

$$l_x = \xi_x \cap T_x\Sigma$$

For most x, the subspace l_x will be a line in $T_x\Sigma$, but at some points, which we call singular points, $l_x = T_x\Sigma$.

We can find a singular foliation F of Σ tangent to l_x at each x, i.e. the complement of the singularities is the disjoint union of 1-manifolds, called leaves of F, and the leaf through x is tangent to l_x. This singular foliation is called the characteristic foliation.

We talk about foliations in greater detail in Chapter 4.

Example 2.3.1. Characteristic foliation on S^2

Let Σ be the unit sphere in R^3, ξ where ξ is the radially symmetric contact structure. The only singularities of this characteristic foliation are at the poles.

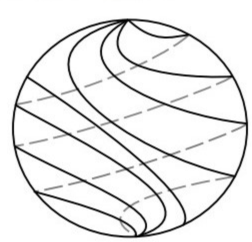


Figure 2.4: Characteristic foliation of S^2 [4]

[4] Etnyre, arXiv:math/0111118v2

Any surface may be perturbed by a C^∞-small isotopy so that its characteristic foliation has only “generic” isolated singularities. A singularity is called “generic” if it looks like one of the following (Figure 2.1). The one on the left is called an elliptic singularity and the on one the right hand side is a hyperbolic singularity.

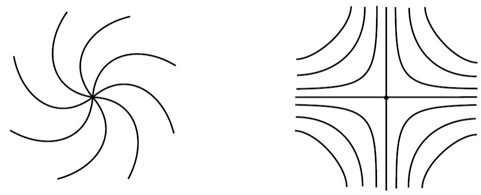


Figure 2.5: Generic singularities of characteristic foliation [5]

Example 2.3.2. (Overtwisted Disk)

Let Σ be the disk of radius π in $r\theta$-plane in $(\mathbb{R}^3, \xi_3)$. Then the characteristic foliation is as shown in 2.6,

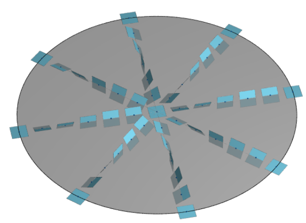


Figure 2.6: Overtwisted disk [6]

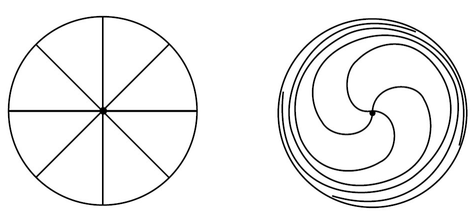


Figure 2.7: Unperturbed and perturbed (only generic isolated singularities) characteristic foliation on D^2 [8]

[5]Etnyre, arXiv:math/0111118v2
[6]Figure courtesy of P. Massot
[8]Etnyre, arXiv:math/0111118v2

Orienting the characteristic foliation: Since our surface Σ is oriented we can choose an orientation for the contact structure ξ. We can orient l_x as; the vector $v \in l_x$ orients it if for vectors $v_\xi \in \xi_x$ and $v_\Sigma \in T_x\Sigma$ such that (v, v_ξ) orients ξ_x and (v, v_Σ) orients $T_x\Sigma$, (v, v_ξ, v_Σ) orients M. We assign a sign to each singular point based on whether the orientation of the contact plane at that point agrees (positive) or disagrees (negative) with the orientation of the tangent plane at that point.

Example 2.3.3. (Singularities of characteristic foliation of S^2) In Example 2.3.1, The singularities at the poles are elliptic points, with one the one at the top a positive singularity (source) and the one below a negative one (sink).

Theorem 2.3.4. *Let (M_1, ξ_1) and (M_2, ξ_2) be contact manifolds and Σ_1 and Σ_2 embedded surfaces. If there is a diffeomorphism $f : \Sigma_1 \to \Sigma_2$ that preserves the characteristic foliation, i.e., $f(F_{\Sigma_1}) = F_{\Sigma_2}$ then f can be extended to a contactomorphism in some neighbourhood of Σ_1.*

2.4 Tight and overtwisted contact structures

A contact structure ξ on M is called *overtwisted* if there is an overtwisted disc present i.e. an embedded disk D whose characteristic foliation is homeomorphic to either of the ones shown in Example 2.9. A contact structure is called *tight* if it does not contain an overtwisted disk.

Lemma 2.4.1. *The existence of overtwisted disk is a contactomorphism invariant.*

This is a non-local invariant that can distinguish contactomorphism classes.

2.5 Knots in Contact manifold

There are two main types of knots in contact 3-manifolds: Legendrian and Transverse. These are interesting on their own, such as their classification problems.

Legendrian and Transverse knots

Definition 2.5.1. A Legendrian Knot in a contact 3-manifold (M, ξ) is an embedding $\gamma : S^1 \to M$, which satisfies $\gamma'(\theta) \in \xi_{\gamma(\theta)}$ for all $\theta \in S^1$

Definition 2.5.2. A Transverse Knot in a contact 3-manifold (M, ξ) is an embedding $\gamma : S^1 \to M$, which satisfies $\gamma'(\theta) \notin \xi_{\gamma(\theta)}$ for all $\theta \in S^1$

Theorem 2.5.3. *([9]) Any knot be in a contact 3-manifold can be $\mathcal{C}^0$ approximated by a Legendrian Knot.*

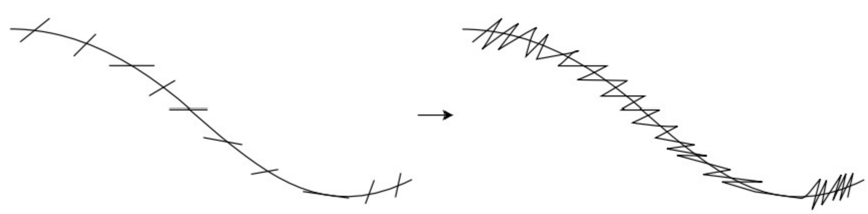


Figure 2.8: $\mathcal{C}^0$ approximation by a Legendrian curve [9]

We can associate a Transverse knot or another Legendrian knot to a given knot, known as the Transverse push-off and Legendrian push-off respectively. These, knots are defined up to Transverse or Legendrian isotopy. These helps us translate results proven for one to the other.

Theorem 2.5.4. *Legendrian Knots up to positive stabilization and Legendrian isotopy has a one-to-one correspondence with tranverse knots upto Transverse isotopy.*

Theorem 2.5.5. *Given any topological knot there is a Legendrian knot C^∞ close to it.*

Front and Lagrangian projections

In order to visualize knots in R^3 we look at their projections onto planes. In contact geometric setting two useful projections of a Legendrian knot are as follows,

[9]Adapted from Lecture notes of Ko Honda

Definition 2.5.6. The front projection of a parametrized knot $\gamma(t) = (x(t), y(t), z(t)$ in $(\mathbb{R}^3, \xi_{std})$ is the projection to xz-plane given by,

$$\gamma_F(t) = (x(t), z(t)).$$

its Lagrangian projection is given by projection to xy plane given by,

$$\gamma_L(t) = (x(t), y(t)).$$

Front projections determine the Legendrian knot upto isotopy as the $y = \frac{dz}{dx}$ using the contact condition. Similarily, given a Lagrangian projection we can determine the knot upto isotopy and z-translation by $z(t_0) = z(0) + \int_0^{t_0} y(t)x'(t)dt$

Theorem 2.5.7. *Any two Legendrian knots are isotopic to each other if and only if their front projections are related by a series of Legendrian Reidemeister moves and planar isotopy preserving cusps and avoiding vertical tangents.*

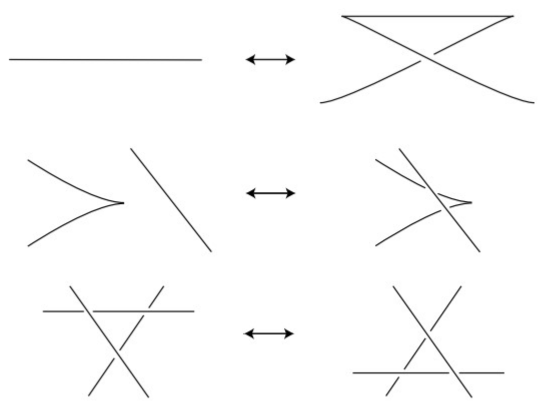


Figure 2.9: Legendrian Reidemeister moves [10]

2.5.1 Classical Invariants of Legendrian and Transverse Knots

In order to classify such knots in R^3, a first step would be to look for invariants. For Legendrian knots we have the following two easily defined invariants.

[10] Adapted from Lecture notes of Ko Honda

Thurston Bennequin Number

Definition 2.5.8. The Thurston Bennequin number, denoted $tb(K)$, is the twisting of the contact framing (trivialization of the normal bundle) with respect to the Seifert surface of K, with right-handed twists being counted positively.

If γ is a Legendrian Knot and σ the surface bounded by it, consider a vector field v transverse to ξ along K, and let the Transverse push-off of K along v be K'. The signed intersection number of K' with Σ is the Thurston-Bennequin number.

It can be computed from the front projection as follows,

$$tb(K) = writhe(K_F) - \frac{1}{2}\#(cusps(K_F))$$

It is also equal to the writhe of its Lagrangian projection.

Rotation Number

Definition 2.5.9. Let Σ be a Seifert surface for K, then the rotation number $rot(K, c)$ counts the number the number of rotations of the positive tangent vector to K relative to the trivialization of ξ_σ over Σ.

If we choose a vector field v along an oriented Legendrian knot such that it induces the orientation, then rotation number can be thought of as the obstruction to extending v to a non-zero vector field in ξ_Σ. It does not depend on the choice of the trivialization.

It can be computed from the front projection as follows,

$$r(K) = \frac{1}{2}(D - U)$$

where D is the number of up cusps and U the number of down cusps.

Figure 2.10 depicts some examples of Knots and their Thurston-Bennequin and Rotation numbers.

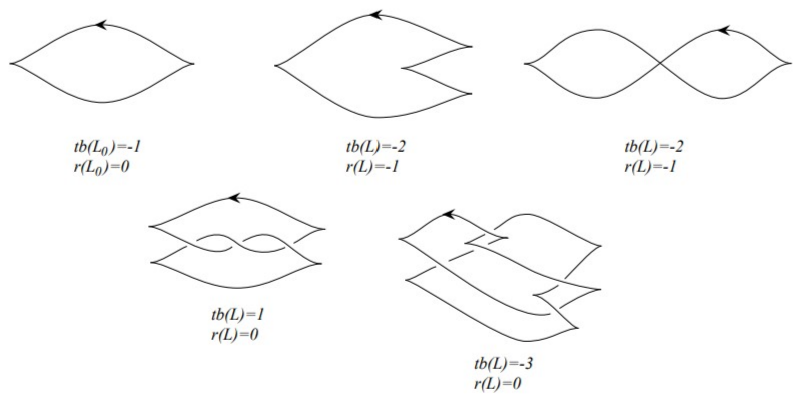


Figure 2.10: Knots with their Thurston-Bennequin and Rotation numbers.[11]

2.5.2 The Bennequin Inequality

The following inequality is an important result towards the characterization of contact structures by analyzing knots in a contact 3-manifold.

Theorem 2.5.10. *(Eliashberg, [6]) Let T be a Transverse knot in a tight contact 3-manifold (M, ξ) with Seifert surface Σ_T. Then,*

$$sl(T) \leq -\chi(\Sigma_T)$$

where $sl(T)$ denotes the self linking number of T and $\chi(\Sigma_T)$ is the Euler characteristic of Σ_T.

Proof. We orient σ_T such that T is its oriented boundary. With this orientation the characteristic foliation of σ_T points outwards from the boundary. We perturb σ_T so that the characteristic foliation is generic. and we assume that the singularities are isolated elliptic or hyperbolic points. Let $e\pm$ be the number of $\pm$ elliptic singularities in σ_{T_ξ} and let $h\pm$ be the number of $\pm$ hyperbolic singularities. If we think of $sl(T)$ as a relative Euler class i.e., let v be a vector field along T tangent to σ_T and contained in ξ and pointing into σ_T , then $sl(T)$ is the obstruction to extending v to a nonzero vector field on σ_T . Then we get,

$$-sl(T) = (e_+ - h_+) - (e_- - h_-)$$

[11]Adapted from Lecture notes of Ko Honda

Next by thinking of building our surface out of discs and bands attached to it, where each disc contains an elliptic point and each band a hyperbolic point, the Euler characteristic of the surface can be expressed as,

$$\chi\left(\Sigma_T\right) = \left(e_+ + e_-\right) - \left(h_+ + h_-\right)$$

Adding both the above equations we get,

$$sl(T) + \chi\left(\Sigma_T\right) = 2\left(e_- - h_-\right)$$

Now since we have a tight contact manifold, every positive or negative elliptic point is connected to a positive or negative hyperbolic point respectively, we can use the Elimination Lemma (Giroux, Fuchs, [7]) to cancel them pairwise 2.11. Thus we can isotope our surface such that the number of elliptic points are zero [6]. Then our above equation becomes,

$$sl(T) + \chi\left(\Sigma_T\right) = 2\left(-h_-\right) \leq 0$$

□

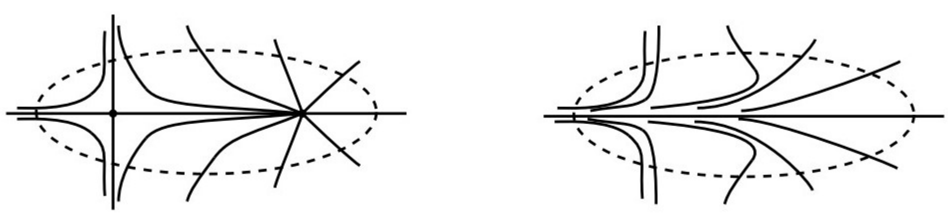


Figure 2.11: Elimination Lemma : Cancellation of singularities pairwise. [12]

[12]Etnyre, arXiv:math/0111118v2

Chapter 3

Open Books

To understand contact structures on 3-manifolds we would like to understand an important way to build or decompose a 3-manifold - Open Book Decomposition. This serves as a key tool in our discussions and proofs in Chapter 5. We define Open Books,look at examples of it, discuss various construction - ways to decompose 3-manifolds into Open Books, construct new Open Books. We also explore connections of Open Books to contact structures in details which will serve as basis for understanding the next few chapters.

3.1 Open Book Decomposition

Definition 3.1.1. An Open Book decomposition of a closed oriented 3 manifold M is a pair (L, π) where

1. L is an oriented link in M called the binding of the Open Book and
2. $\pi : M \backslash L \to S^1$ is a fibration of the complement of B such that $\pi^{-1}(\theta)$ is the interior of a compact surface $\Sigma_\theta \subset M$ and $\partial \Sigma_\theta = L$ for all $\theta \in S^1$. The surface $\Sigma = \Sigma_\theta$ for any θ, is called the page of the Open Book.

Given an Open Book we can describe $M \backslash L$ as the mapping cylinder of diffeomorphism $\phi : \Sigma \to \Sigma$ of a surface Σ. We can recover M and the Open Book (L, π), up to diffeomorphism, from the pair (Σ, ϕ). ϕ is called the monodromy of the Open Book. We can formalize these observations in the definition of an abstract Open Book.

Definition 3.1.2. An abstract Open Book is a pair (Σ, ϕ) where,

1. Σ is an oriented compact surface with boundary and
2. $\phi : \Sigma \rightarrow \Sigma$ is a diffeomorphism such that ϕ is the identity in a neighborhood.

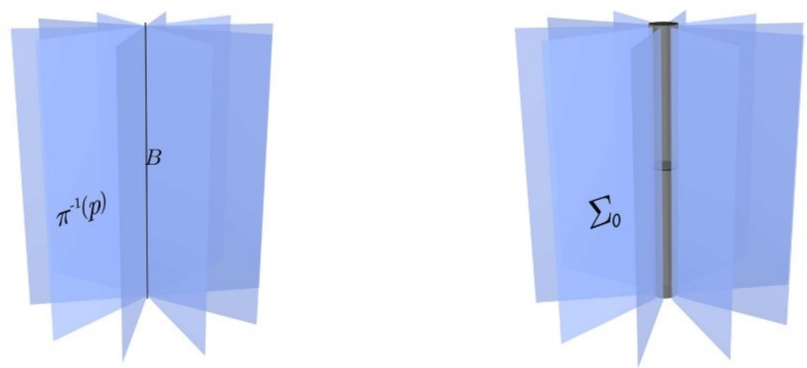


Figure 3.1: An Open Book (left) and an abstract Open Book (right) [1]

3.1.1 Building 3-manifolds from Abstract Open Book

A natural way to construct a 3-manifold is as a mapping torus of a diffeomorphism of a surface. That is, we take a surface $\Sigma \times [0, 1]$ and glue it up using a diffeomorphism of the surface. This way of constructing 3-manifolds enable us to find an abstract Open Book decomposition of it.

Conversely,we can build a 3-manifold M_ϕ starting from an abstract Open Book in the following manner. Start by considering the mapping torus Σ_ϕ of ϕ, Σ x [0,1] / $\sim$, where $\sim$ is the equivalence relation $(\phi(x), 0) \sim (x, 1)$ for all $x \in \Sigma$. We then glue in a copy of $D^2 \times S^1$ along each component of the boundary such that $\{\text{pt}\}$ x $S^1 \subset \delta\Sigma \times S^1$ bounds a disc. We can write this as,

[1]Andy Wand, Surgery and tightness in contact 3-manifolds, Proceedings of 21st Gökova, Geometry-Topology Conference,pp. 234 – 249

$$M_\phi = \Sigma_\phi \cup_\psi \left(\coprod_{|\partial\Sigma|} S^1 \times D^2 \right)$$

where $|\delta\Sigma|$ denotes number of boundary components, $\coprod_{|\partial\Sigma|}$ denotes that the diffeomorphism ψ is used to identify the boundaries. The cores of the solid tori are denoted by B_ϕ which gives us the binding of an Open Book.

Two abstract Open Books (Σ_1, ϕ_1) and (Σ_2, ϕ_2) are called equivalent if there is a diffeomorphism $h : \Sigma_1 \to \Sigma_2$ such that $h \circ \phi_2 = \phi_1 \circ h$.

Lemma 3.1.3. *We have the following relations between Open Books and abstract Open Books:*

1. *An Open Book decomposition (B, π) of M gives an abstract Open Book (Σ_π, ϕ_π) such that $(M_{\phi_\pi}, B_{\phi_\pi})$ is diffeomorphic to (M, B).*
2. *An abstract Open Book determines M_ϕ and an Open Book (B_ϕ, π_ϕ) up to diffeomorphism.*
3. *Equivalent Open Books give diffeomorphic 3-manifolds.*

Clearly the two notions of Open Book decomposition are closely related. The basic difference is that in the case of Open Books (non-abstract) we can discuss the binding and pages up to isotopy in M, whereas for abstract Open Books we can only describe them up to diffeomorphism.

3.1.2 Examples

Example 3.1.4. (Open Book decomposition of S^3 [10]) Let S^3 be the unit sphere in $\mathbb{C}^2$ and let $(z_1, z_2) = (r_1 e^{\iota\theta_1}, r_2 e^{\iota\theta_2})$ be coordinates on $\mathbb{C}^2$. Consider an unknotted S^1 sitting in S^3 denoted by $U = \{z_1 = 0\}$ and the fibration of the complement given by,

$$\pi_U : S^3 \backslash U \to S^1 : \left(r_1 e^{\iota\theta_1}, r_2 e^{\iota\theta_2}\right) \mapsto \theta_1$$

This Open Book decomposition is corresponds to the decomposition of S^3 into two solid tori.

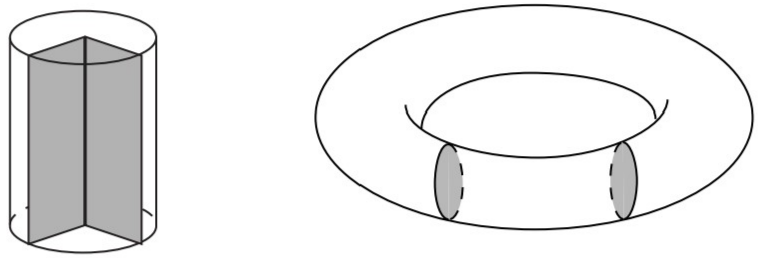


Figure 3.2: Decomposition of S^3 into two solid tori [10]

Example 3.1.5. (Abstract Open Book decomposition of S^3) Let $\Sigma = D^2$ be a disc and $\phi : \Sigma \to \Sigma$ be the identity map. Then the pair (D^2, id) gives an abstract Open Book for S^3. This is diffeomorphic to the above as it decomposes S^3 into two tori, the one given by $D^2 \times S^1$ and the other given by the solid torus we glue in along the boundary.

Example 3.1.6. (Another Open Book for S^3) Consider the positive and negative Hopf link in S^3 given $H^+ = \{(z_1, z_2) \in S^3 : z_1 z_2 = 0\}$ and $H^- = \{(z_1, z_2) \in S^3 : z_1 \overline{z_2} = 0\}$ respectively.

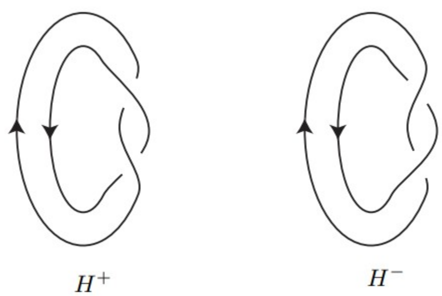


Figure 3.3: Positive and negative Hoph links[10]

We have the following fibrations for the knot complements,

$$\pi_+ : S^3 \backslash H^+ \to S^1 : (z_1, z_2) \mapsto \frac{z_1 z_2}{|z_1 z_2|}, \text{ and}$$
$$\pi_- : S^3 \backslash H^- \to S^1 : (z_1, z_2) \mapsto \frac{z_1 \overline{z_2}}{|z_1 \overline{z_2}|}$$

In polar coordinates these maps are $\pi_\pm \left(r_1 e^{i\theta_1}, r_2 e^{i\theta_2} \right) = \theta_1 \pm \theta_2$.

Theorem 3.1.7. *(Alexander, [3]) Every closed oriented 3-manifold admits an Open Book decomposition.*

Proof. The proof relies on the following two results,

Lemma 3.1.8. *(Alexander, [3]) Every closed oriented 3-manifold M is a branched cover of S^3 with branched set some link L_M.*

Lemma 3.1.9. *(Alexander, [2]) Every link L in S^3 can be braided about the unknot.*

A link L is said to be braided about the unknot U if for $S^1 \times D^2 = S^3 \backslash U$ we can isotop L such that $L \subset S^1 \times D^2$ and L is Transverse to $\{p\} \times D^2$ for all $\{p\} \in S^1$.

Now using the first lemma, given a closed oriented 3-manifold M, we have a link $L_M \subset S^3$. By the second lemma we can braid L_M about the unknot U. Let $P : M \to S^3$ be the branched covering map. Now we set $B = P^{-1}(U) \subset M$. We claim that B is the binding of an Open Book. B is an oriented link in M as P is a branched covering map and L_M does not intersect U. The fibering of the complement of B is given by $\pi = \pi_U \circ P$, where π_U is the fibering of the complement of U in S^3. Since $P^{-1}(U) = B$ and $\pi_U : S^3 \to S^1$, $\pi : M \backslash B \to S^1$ is well defined. Since π_U is the fibering of the complement of U in S^3, $\pi_u^{-1}(\theta)$ is the interior of a compact surface in S^3 with boundary as U, $\pi^{-1}(\theta)$ is also a compact surface with boundary as B. □

3.1.3 Building new Open Books from existing ones

In the above examples we constructed Open Books for manifolds by finding an explicit fibration of a link complement. But this can be difficult to do for more complicated manifolds. The following result helps us build new Open Books from existing ones, letting us find Open Book decompositions of manifolds built out of other simpler manifolds.

Definition 3.1.10. Given two abstract Open Books (Σ_i, ϕ_i), $i = 0, 1$, let c_i be an arc properly embedded in Σ_i and R_i a rectangular neighborhood of c_i, $R_i = c_i \times [-1, 1]$. The Murasugi sum of two Open Books, (Σ_0, ϕ_0) and (Σ_1, ϕ_1) is the Open Book $(\Sigma_0, \phi_0) * (\Sigma_1, \phi_1)$ with page

$$\Sigma_0 * \Sigma_1 = \Sigma_0 \cup_{R_1 = R_2} \Sigma_1,$$

where R_0 and R_1 are identified so that $c_i \times \{-1, 1\} = (\delta c_{i+1}) \times [-1, 1]$, and the monodromy is $\phi_0 \circ \phi_1$.

Theorem 3.1.11. *(Gabai, [12]) $M_{(\Sigma_0,\phi_0)} \# M_{(\Sigma_1,\phi_1)}$ is diffeomorphic to $M_{(\Sigma_0,\phi_0)*(\Sigma_1,\phi_1)}$.*

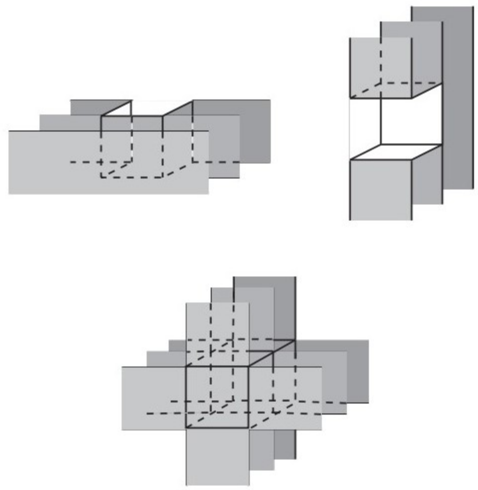


Figure 3.4: At the top left is a piece of $\Sigma_0 \times [0,1]$ with B_0 cut out. The lightest shaded part is $\Sigma_0 \times \{0\}$ the medium shaded part is $\Sigma_0 \times \{\frac{1}{2}\}$ and the darkest shaded part is $\Sigma_0 \times \{1\}$. The top right is a similar picture for Σ_1. The bottom picture is $(\Sigma_0 * \Sigma_1) \times [0,1]$ [10]

Proof. Let $B_0 = R_0 \times [\frac{1}{2}, 1]$ be a 3-ball in $M_{(\Sigma_0,\phi_0)}$ and $B_1 = R_1 \times [0, \frac{1}{2}]$ be a 3-ball in $M_{(\Sigma_1,\phi_1)}$. We can form $(\Sigma_0 * \Sigma_1) \times [0,1]$ as shown in the Figure 3.4 .

For constructing the mapping cylinder of ϕ_0 we think of gluing $\Sigma_0 \times \{0\}$ to $\Sigma_0 \times \{1\}$ using the identity and then cutting the resulting $\Sigma_0 \times S^1$ along $\Sigma_0 \times \{\frac{1}{4}\}$ and regluing using ϕ_0. Similarly for the mapping cylinder of ϕ_1 we perform a regluing of $\Sigma_1 \times S^1$ along $\Sigma_1 \times \{\frac{3}{4}\}$. Note that this construction avoids B_0 and B_1. We construct the mapping cylinder for $\phi_0 \circ \phi_1$ in a similar fashion by regluing $(\Sigma_0 * \Sigma_1) \times S^1$ by ϕ_0 along $(\Sigma_0 * \Sigma_1) \times \{\frac{1}{4}\}$ and by ϕ_1 along $(\Sigma_0 * \Sigma_1) \times \{\frac{3}{4}\}$. Thus we see the mapping cylinders fit together nicely with the pages and the monodromy of both sides matching up. The binding also fits properly as the number of boundary components in $\Sigma_0 * \Sigma_1$ is the sum of boundary components in Σ_0 and Σ_1.

$\square$

3.1.4 Stabilization of Open Books

Definition 3.1.12. A positive (negative) stabilization of an Abstract Open Book (Σ, ϕ) is the Open Book

2. monodromy $\phi\prime = \phi \circ \tau_c$ where τ_c is a right-(left-)handed Dehn twist along a curve c in $\Sigma\prime$ that intersects the co-core of the 1-handle exactly one time. We denote this stabilization by $S_{(a,\pm)}(\Sigma, \phi)$ where $a = c \cap \Sigma$ and $\pm$ refers to the positivity or negativity of the stabilization.

Example 3.1.13. (Stabilization of (D^2, id) which is related to the Hopf link Open Book) Consider the Abstract Open Book (D^2, id) for S^3. Upon stabilization, attaching a 1-handle results in the new page Σ' which an annulus and the monodromy $\phi' = id \circ \tau_c$ is a right-handed Dehn twist. Thus $S_\pm(D^2, id) = (A^2, \tau_c)$, which is the abstract Open Book for S^3 corresponding to the Open Book with binding as the positive (negative) Hopf link.

Lemma 3.1.14.

$$S_\pm(\Sigma, \phi) = (\Sigma, \phi) * \left(H^\pm, \pi_\pm\right)$$

where $H^\pm$ is the positive/negative Hopf link and $\pi_\pm$ is the corresponding fibration of its complement.

Proof. From the above example we know that the abstract Open Book corresponding to positve (negative) Hopf link has annular pages and the monodromy is given by positve (negative) Dehn twist. By the definition of Murasugi sum, pages of $(\Sigma, \phi) * (H^\pm, \pi_\pm)$ are given by $\Sigma' = \Sigma \cup$ 1-handle, since the only properly embedded curves of A^2 are as in the figure below and identifying a rectangle about it with a rectangle about a properly embedded curve in Σ is the same as attaching a 1-handle . The monodromy of the resultant abstract Open Book is $\phi\prime = \phi \circ \tau_c$. This is exactly the same as doing a stabilization.

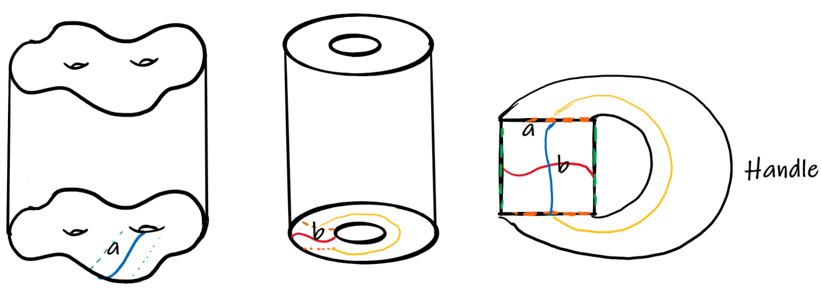


Figure 3.5: From left, Open Book (Σ, ϕ), Open Book (A^2, τ), Identifying pages along a rectangle

□

Corollary 3.1.15. $M_{(S\pm(\Sigma,\phi))} = M_{(\Sigma,\phi)}$

Proof. This follows from Theorem 3.1.11 and the above lemma along with the fact that $M_{(H^{\pm}}, \pi_{\pm}) = S^3$. □

Theorem 3.1.16. *Every 3-manifold admits an Open Book decomposition with connected binding.*

Proof. Consider an abstract Open Book, then the binding corresponds to the core of the solid tori attached to the boundary for each boundary component. Suppose the binding is not connected. Then we consider the boundary components corresponding to two different components and take an embedded curve connecting both and perform a stabilization about it. This results in connecting the two components. We can do this for all disjoint components. □

3.2 Open Books And Contact Structures

A contact structure ξ on M is said to be supported by an Open Book (B, π) if ξ can be isotoped through contact structures so that there is a contact 1-form α for ξ such that

1. $d\alpha$ is a positive area form on each page Σ_θ of the Open Book and
2. $\alpha > 0$ on the tangent to the oriented binding B.

Theorem 3.2.1. *(Thurston, Winkelnkemper [16]) Every Open Book decomposition* (Σ, ϕ) *supports a contact structure* ξ_ϕ

Proof. Let

$$M_\phi = \Sigma_\phi \cup_\psi \left(\coprod_{|\partial\Sigma|} S^1 \times D^2 \right)$$

be as before where Σ_{phi} is the mapping torus of ϕ. We first construct a contact structure on $\Sigma \times [0,1]/\sim$ and then extend it in a neighbourhood of the binding.

Consider coordinates (Ψ, r, θ) in the neighbourhood of the binding of each component, such that (Ψ, r) are coordinates along the page with Ψ along the binding and $d\theta$ and $\pi^* d\theta$ agree, where $\pi : M \backslash L \to S^1$ and θ is the coordinate along S^1. Let λ be a 1-form on the page which is an element of the set,

$$\begin{aligned} S = \{1 \text{ -forms } \lambda : \quad & (1)\lambda = (1+r)d\theta \text{ near } \partial\Sigma \text{ and} \\ & (2)d\lambda \text{ is a volume form on } \Sigma\} \end{aligned}$$

To check that this set is non-empty let λ_1 be a 1-form that has the right form near the boundary. Now,

$$\int_\Sigma d\lambda_1 = \int_{\partial\Sigma} \lambda_1 = 2\pi|\partial\Sigma|$$

Let ω be a volume form on Σ such that it is $dx \wedge d\theta$ near the boundary and its integral over Σ is $2\pi|\partial\sigma|$. Then,

$$\int_\Sigma (\omega - d\lambda_1) = 0$$

and $\omega - d\lambda_1 = 0$ near the boundary. By de Rham theorem we can now find a 1-form β that vanishes near the boundary and $d\beta = \omega - d\lambda_1$. Then, $\lambda = \lambda_1 + \beta$ is an element of S.

Now for a 1-form λ is S, $\phi^*\lambda$ also belongs to S. Consider the 1-form $\tilde{\lambda} = (1-\theta)\lambda + \theta(\phi^*\lambda)$ on $\Sigma \times [0,1]$ and let $\alpha_K = \tilde{\lambda} + Kd\theta$. For sufficiently large K, α_K is a contact form and it descends to a contact form on $Sigma \times [0,1]/\sim$. Next we extend this form to the solid tori neighbourhood of the binding by pulling back α through the gluing map f to get,

$$\alpha_f = Kd\theta - (r+\epsilon)d\psi$$

. To extend this form on the entire $S^1 \times D^2$ to a contact form of the form $h(r)d\psi + g(x)d\theta$. For this we need functions $h(r), g(r) : [0,1] \to R^3$ satisfying:

1. $h(r)g'(r) - h'(r)g(r) > 0$ (contact condition)
2. $h(r) = 1$ near $r = 0$ and $h(r) = -(r+\epsilon)$ near $r = 1$.
3. $g(r) = r^2$ near $r = 0$ and $g(r) = K$ near $r = 1$.

If we define the functions as in the Figure 3.6 below it satisfies all our conditions assuming δ_h and δ_g are such that $\delta_h < \delta_g$ and $h(r) < 0$ on $[\delta_h, 1]$ and $g(r) = K$ on $[\delta_g, 1]$.

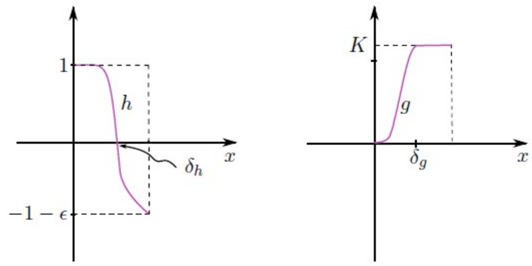


Figure 3.6: h and g functions

□

Furthermore it was observed by Giroux that this contact structure is unique up to isotopy.

Theorem 3.2.2. *(Giroux, [14]) Two contact structures supported by the same Open Book are isotopic.*

Proof. Let ξ_0 and ξ_1 be two contact structures supported by the same Open Book (B, π) and α_0 and α_1 the corresponding 1-forms. We now construct contact forms α_{0R} and α_{1R} as follows.

Let $d\theta$ be the coordinate along S^1, where $M \backslash B$ fibers over S^1. Let $f : [0, \epsilon] \to R$ be an increasing non-negative function that equals r^2 near 0 and 1 near ϵ and beyond it, where ϵ such that $r < \epsilon$ in the solid tori neighbourhood of the binding. Now consider the 1-form

$$\alpha_{0R} = \alpha_0 + R_0 f(r) d\theta$$

where R_0 is some large constant. It can be easily checked that this is a contact form since α_0 is a contact form and positive on the binding, $d\alpha_0$ a volume form on the pages. We can similarly construct α_{1R}. Now consider the form,

$$\alpha_t = t\alpha_{1R} + (1-t)\alpha_{0R}$$

We can verify that α_t is a contact form for large R for all $0 \le t \le 1$.

□

The following is a central result in contact geometry that relates contact structures to Open Book decompositions.

Theorem 3.2.3. *(Giroux, [14]) Let M be a closed oriented 3-manifold. Then there is a one to one correspondence between*

$$\{\textit{oriented contact structures on } M \textit{ up to isotopy}\}$$

and

$$\{\textit{Open Book decompositions of } M \textit{ up to positive stabilization}\}.$$

The results just outlined show that the topological definition of an Open Book succesfully captures the apriori geometric data of a contact structure. We will see various applications of this theorem in the following chapters.

Chapter 4

Plane Fields

Our discussion of contact structures in the previous chapter tells us that a 2-dimensional plane field on a 3-manifold is not necessarily integrable, even locally. Infact,non-integrable plane fields give us contact structures. This leads to the question of what other plane field structures are possible on a 3-manifold. In this chapter we explore foliations, confoliations, contact structures and their relations.

4.1 Foliations, Contact Structures and Confoliations

A foliation of a manifold is a particular type of decomposition into submanifolds. Characteristic foliations, which were discussed in Chapter 1, offer an example of a singular foliation of a surface. In the rest of the chapter we look at foliations by surfaces of a 3-manifold.

Definition 4.1.1. A co-dimension 1 foliation of the 3-manifold is an integrable plane field ξ. Equivalently, it is a decomposition into surfaces such that each point has a neighbourhood $D^2 \times I$ on which integrating ξ yields a fibration by horizontal discs $D^2 \times t$, $t \in I$.

The surfaces formed by integrating ξ are known as leaves of the foliation.

For the rest of our discussion we assume ξ is co-orientable or transversely oriented i.e., the normal bundle to the plane field is orientable.

Note: In general we do not distinguish between the plane field tangent to a foliation and the foliation itself and refer to an integrable plane field as foliation too.

If a plane field ξ is defined (locally, for ξ coorientable) as the kernel of a 1-form α, then by the Frobenius Theorem the equation

$$\alpha \wedge d\alpha = 0$$

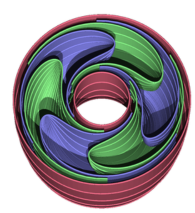
is a sufficient and necessary condition for integrability.

Definition 4.1.2. A plane field $\xi = \ker \alpha$ on an oriented manifold is called a positive (negative) confoliation if $\alpha \wedge d\alpha \geq 0$ ($\alpha \wedge d\alpha \leq 0$).

Foliations and contact structures thus lie at the different extremes on the scale of confoliations.

The following is an example of a foliation of the torus. We use this in the proof of Theorem 5.2.2.

Example 4.1.3. A Reeb component is a foliation of a solid torus (D^2 x S^1) by planar leaves. The unique compact leaf is $D^2 \times S^1$. Each non-compact leaf can be thought of as a D^2 folded like a hemisphere with boundary asymptotically tending to the boundary of the torus (like a sock) which is swallowed by the next disk shaped similarly.



4.2 Taut vs Tight

A contact structure on a 3-manifold is said to be tight if it is not overtwisted i.e., there is no overtwisted disc present in the contact manifold.

Definition 4.2.1. On the other hand a foliation is said to be taut if it is not the foliation of $S^2 \times S^1$ by spheres $S^2 \times p$, $p \in S^1$ and satisfies any one of the following equivalent conditions (the equivalence is due to Novikov, Sullivan [17]):

1. each leaf of the foliation is intersected by a tranversal closed curve
2. there exists a vector field which is transversal to the foliation and preserves a volume form on a manifold.
3. the manifold admits a Riemannian metric for which all leaves are minimal surfaces.

A taut foliation cannot have Reeb components. This is a necessary and sufficient condition. The presence of a Reeb component violates the existense of a transversal closed curve intersecting each leaf as it acts as a dead end, a transversal curve encountering it cannot escape from it.

4.2.1 Tight and Taut confoliations

The notion of tightness and tautness can be generalised to confoliations.

Definition 4.2.2. A confoliation (M, ξ) is tight if for every embedded 2-disk $D \subset M$ satisfying

- ∂D is tangent to ξ
- D is transversal to ξ in a neighbourhood of the boundary.

there is another disk D' such that

- The boundary of D and D' agree
- D' is tangent to ξ
- $e(\xi)[D \cup D'] = 0$

Let us see that definition coincides with the definition of tight for ξ a contact structures, and Reebless in case of ξ a foliation.

In the contact case this coincidence is straightforward. In case of a contact structure, we cannot have such a D' that is tangent everywhere as it violates the contact condition, hence, we cannot have such a D (overtwisted). The absence of overtwisted disc implies the contact structure is tight.

The foliation case is more involved. For a foliation, the above definition of tight is equivalent to the absence of vanishing cycles, which in turn is equivalent to the absence of Reeb components for a closed 3-manifold.

Definition 4.2.3. (Vanishing Cycle) A closed path (loop) $\gamma : [0, 1] \to L_z$ (a leaf containing z) is called a vanishing cycle if,

- γ is not homotopic to zero in L_z.
- There exists, a sequence of points $z_n \to z$ and a sequence of loops $\gamma_n : [0, 1] \to L_{z_n}$ such that γ_n converges uniformly to γ and γ_n is homotopic to zero in L_{z_n}.

Proof. We begin by observing that presence of a Reeb component implies existence of vanishing. Let γ be a meridional loop on the outer torus leaf through a point z on it, this is clearly not homotopic to zero on the torus leaf. Consider the disc with γ as boundary and let its intersection with the inner leaves be γ_n with n increasing as we move away towards the boundary, where z_n lie on the radial line joining the centre of the disc to z. This forms our vanishing cycle.

Since a Reeb component implies the presence of a vanishing cycle, the absence of a vanishing cycle implies no Reeb component. This proves one direction of the above claim.

To show that Definition 4.3.1 implies absence of vanishing cycle, suppose γ is a vanishing cycle in L_z. Then let D be the disc with γ as boundary. Since γ is not homotopic to zero in L_z, $D \not\subset L_z$. Now by definition there exists a disc D' such that $TD' \in \xi$. Therefore $D' \subset L_z$. Therefore γ is homotopic to zero in L_z. This gives us a contradiction. □

4.3 Perturbing confoliations to Contact Structures

In this section we explore the possibilities of perturbing a foliation or a confoliation to a contact structure. There are three different ways in which we can achieve this pertubation.

A deformation is a transformation of a confoliation to a contact structure via a path through plane fields that are contact at all points on the path except at the confoliation end.

A confoliation $\xi = \ker\alpha$ is said to be linearly deformed to a contact structure if there is a deformation $\xi_t = \ker\alpha_t$, $t \in R^+$, such that $\alpha_0 = \alpha$ and

$$\frac{d(d\alpha_t \wedge \alpha_t)}{dt}|_{t=0} > 0$$

A confoliation $\xi = \ker\alpha$ is said to be C^k deformed in to a contact structure if there is a C^k deformation starting at ξ.

In case the perturbation is not a deformation we call it an C^k approximation.

Holonomy

Let $(M, \mathcal{F})$ be a foliation and L be a leaf. For a path $\gamma : [0,1] \to L$ contained in the intersection of the leaf with a foliation chart, and two transversals τ_0, τ_1 to γ at the endpoints, the product structure of the foliation chart determines a homeomorphism

$$h : \tau_0|_u \to \tau_1|_u$$

.

Lemma 4.3.1. *Let $(M, \mathcal{F})$ be a foliation and L be a leaf, $x \in L$ and τ a transversal to x. Holonomy transport defines a homeomorphism*

$$H : \pi_1(L, x) \to \text{Homeo}(\tau)$$

to the group of germs of homeomorphism of τ.

A foliation with no holonomy refers to a foliation all of whose leaves have trivial holonomy.

A closed 1-form without singularity is on a manifold is integrable, hence defines a codimension-1 foliation on it. In particular any foliation defined by a closed 1-form is without holonomy.

Lemma 4.3.2. *Two homotopic paths with same endpoints induce same holonomy.*

However, the converse doesn't hold.

Computing the Holonomy group

Let $\mathcal{F}$ be a codimension-q foliation on M and $p \in M$. We choose an embedding $\psi : D^q \to M$ such that $\psi(0) = p$ and it is transverse to the foliation. To compute the holonomy group we associate a germ of a diffeomorphism h_γ of a neighbourhood of 0 in D^q to another neighbourhood of 0 to every oriented loop γ based at p and contained in the leaf containing p. We do this as follows,

We choose a transverse map $\Psi : D^q \times S^1 \to M$ satisfying the following,

- $\Psi(o, \theta) = \gamma(\theta)$
- $\Psi|_{D^q \times \{0\} = \psi}$

Now for a point $x \in D^q$, we follow the curve of the foliation induces on $D^q \times S^1$ as θ varies over S^1. This gives us our holonomy map h_γ along γ.

For x sufficiently close to 0 it is possible to return to a new point after passing completely around S^1.

Holonomy of Reeb component

We compute the holonomy group for the unique closed leaf of a Reeb component. By Lemma 4.3.1 it is sufficient to consider the generators of the fundamental group of the torus. Let α and β be the meridional and transverse curves along the torus respectively. To compute the holonomy group we pick a an embedding of D^1 in M, transverse to the foliation (a curve

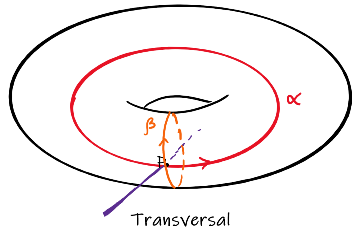


Figure 4.1: Caption

intersecting the boundary torus leaf) transversely. If we now pick a point x slightly away from p and consider the holonomy transport along α it does not return to p. Thus $hol(\alpha)$ generates an infinite group. While if we translate along β we always return to the same point. So $hol(\beta)$ is trivial.

Theorem 4.3.3. *([17]) If a foliation*

- *has a closed leaf with trivial holonomy or*
- *Can be defined by a closed 1-form or*
- *has no holonomy*

then, it can be perturbed into a contact structure.

Theorem 4.3.4. *(Eliashberg and Thurston) A perturbation of a tight foliation is a tight contact structure.*

Theorem 4.3.5. *Every contact structure on a 3-manifold is a deformation of a foliation*

In the next chapter we discuss in detail how to construct such a foliation. In fact, the foliation we construct has a Reeb component.

Corollary 4.3.6. *If ξ is a tight foliation then we can isotop it through tight confoliation to a foliation with Reeb components. In particular a taut foliation is isotopic to a foliation with Reeb components.*

Proof. If we have a tight foliation we can find a contact structure that is a perturbation of it by Theorem 4.3.4. By using the above theorem we can find a foliation with Reeb component that perturbs to this contact structure. Thus composing the two paths we have a path between a tight foliation and one with Reeb component. □

Chapter 5

Contact Structures and Foliations

5.1 Introduction

To elucidate the role of Open Books in the study of plane fields, especially contact structures, we look at the relation between contact structures and codimension one foliations on 3-manifolds. Viewing contact structures as plane fields we can consider foliations in the same homotopy class. Eliashberg and Thurston proved that every foliation can be approximated by contact structures so the question arises as to whether every contact structure is close to a foliation [17]. We can make the notion of closeness precise by defining contact structure ξ to be a deformation of a foliation ζ if there is a one parameter family of plane fields ξ_t such that $\xi_0 = \zeta$ and $\xi_1 = \xi$, and ξ_t is a contact structure for $t > 0$. We can the ask, is every contact structure a deformation of a foliation?

Theorem 5.1.1. *(Etnyre [11]) Every positive and negative contact structure on a closed oriented 3-manifold is a C^∞-deformation of a C^∞-foliation.*

Let us give a careful explanation of Etnyre's proof in the following pages.

The proof draws upon the connections between Open Books and contact structures. The theorem gives rise to various observations and questions regarding the connections between confoliations, foliations and contact structures.

5.2 Open Books and Contact Structures

Before proving the theorem we recall briefly the tools required for the proof, definition of Open Books (Definition 3.1.1) and the Giroux correspondence (Theorem 3.2.3).

A contact structure ξ is said to be supported by an Open Book (L, π) if there is a contact 1-form α such that:

- $\alpha(L) > 0$
- $d\alpha$ is a volume form when restricted to each page.

The Thurston-Winkelnkemper construction allows us to construct a contact structure supported by a given Open Book.

The facts that we use in the proof from this discussion are:

- All contact structures are supported by Open Books.
- The supported contact structure is unique up to isotopy.

5.3 Proof

To prove the Theorem 5.1.1 we follow the following outline,

- We start with a contact structure ξ
- Then we choose some Open Book (L, π) that supports ξ
- We construct a foliation on M associated with the choosen Open Book
- Next we show that we can perturb the foliation into a contact structure supported by our Open Book

- Finally by Giroux correspondence we can conclude that the perturbed contact structure is isotopic to ξ since they are supported by the same Open Book. Thus proving our theorem.

Having chosen a contact structure ξ and our Open Book supporting it (This can always be done), we proceed to constructing our foliation.

5.3.1 Constructing the foliation

The basic idea behind constructing the foliation is to replace the neighbourhoods of the binding by Reeb components (Example 4.1.3) and spinning the pages of the Open Book so that they limit to the Reeb components.

We begin by constructing a foliation on the neighbourhood of the binding and then extending it to the rest. Let N be a neighbourhood of one component of the binding. We choose coordinates (r, θ, ϕ), so that the pages of the Open Book intersecting N correspond to constant θ annuli and the binding corresponds to $r = 0$.

Assume $N = (r, \theta, \phi)|r \leq 1 + 2\epsilon$ for some small fixed ϵ.

We choose two functions $\lambda(r)$ and $\delta(r)$ satisfying the following properties,

$\lambda(r)$ is

- zero on $[0, \frac{1}{3}]$
- one for $r \geq 1$
- strictly increasing on $[\frac{1}{3}, 1]$,

and $\delta(r)$ is,

- zero on $[0, 1]$
- one for $r \geq 1 + \epsilon$

- strictly increasing on $[1, 1+\epsilon]$

Next we set,

$$\alpha = \begin{cases} \lambda(r)dr + (1-\lambda(r))d\phi & \text{for } r \leq 1 \\ \delta(r)d\theta + (1-\delta(r))dr & \text{for } r > 1 \end{cases}$$

Then,

$$d\alpha = \begin{cases} d\lambda \wedge dr - d\lambda \wedge d\phi & \text{for } r \leq 1 \\ d\delta \wedge d\theta - d\delta \wedge dr & \text{for } r > 1 \end{cases}$$

Since, λ and δ are functions of r, $\alpha \wedge d\alpha = 0$. So, $\zeta = ker\alpha$ gives a foliation on N.

Let N_a denotes the set $\{(r, \theta, \phi) | r \leq a\}$ Then subset $N_1 = \{(r, \theta, \phi) | r \leq 1\}$ of N is a Reeb component. We can choose $\lambda(r)$ and $\delta(r)$ such that α defines a C^∞-foliation on N. Now, on the region $[1+\epsilon, 1+2\epsilon] \times T^2$, the foliation is given by constant θ annuli. But constant θ also corresponds to the intersection of the pages with the neighbourhood N, thus this foliation can be extended to the pages, giving us a foliation of M. In particular if dz corresponds to the pull-back of the coordinate on S^1 by the fibration π, then we can extend α by adding dz to get a 1-form defining our foliation on all of M.

5.3.2 Perturbing the foliation

Having constructed a foliation, we may now perturb it into a contact structure supported by the chosen Open Book. We start by looking at the neighbourhood N of L and set,

$$\alpha_t = \alpha + t\left(r^2 d\theta + (1 + f(r))d\phi\right)$$

where $f : N \to \mathbb{R}$ is a strictly decreasing function, $f(0) = 0$, $f(r) > -1$ for all r and $f(r) < -1 + \iota$ for all $r > 1$ and ι some small number.

This is similar to the spherical contact form $r^2 d\theta + dz$ for $r > 1$ as $1 + f(r)$ is small with an additional dr term.

$$d\alpha_t = \begin{cases} (tf'(r) - \lambda'(r))\, dr \wedge d\phi + t2rdr \wedge d\theta & \text{for } r \leq 1 \\ (t2r + \delta'(r))\, dr \wedge d\theta + tf'(r)dr \wedge d\phi & \text{for } r > 1 \end{cases}$$

Thus we have,

$$\alpha \wedge d\alpha =$$

$$\begin{cases} \mathrm{tr}\,(2[(1-\lambda(r)) + t(1+f(r))] - r\,(tf'(r) - \lambda'(r)))\,dr \wedge d\theta \wedge d\phi & \text{for } r \leq 1 \\ t\,(-f'(r)\,(\delta(r) + tr^2) + (1+f(r))\,(t2r + \delta'(r)))\,dr \wedge d\theta \wedge d\phi & \text{for } r > 1 \end{cases}$$

Since, $\lambda(r) < 1$ so $(1-\lambda(r)) > 0$; similarily $f(r) > -1$ implies $(1+f(r)) > 0$; $f'(r) < 0$ and $\lambda'(r) > 0$ for $r \leq 1$; the first term is positive. Similarly the above constraints, along with $\delta'(r) > 0$ for $r > 1$, imply that the second term is positive. Therefore, α_t is a contact form on N for all $t > 0$

Our next step is to construct a family of contact forms on the pages of the Open Book and then patch it with the family of 1-forms on N. To do this recall the Thurston-Winkelnkemper construction (Theorem 3.2.1).

We consider $M \backslash N_{1+\epsilon}$ as the mapping cylinder of ψ (monodromy of our Open Book)

$$M \backslash N_{1+\epsilon} = \Sigma \times [0,1]/(\psi(x), 0) \sim (x, 1)$$

Let the coordinate on the $[0,1]$ factor be z. We then find a 1-parameter family of 1-forms λ_z on Σ so that $d\lambda_z$ is a volume form on Σ for all z and each $\lambda_z = (1+\epsilon+s)$ near each boundary component of Σ, where (s, θ) are polar coordinates near the boundary component and the boundary corresponds to $s = 0$ and s is increasing into Σ. Moreover, the λ_z are chosen so that they descend to give a form on $M \backslash N_{1+\epsilon}$. (the 1-form dz, from before, corresponds to dz in these coordinates. The 1-form $\beta_t = dz + t\lambda z$ will be a contact 1-form on $M \backslash N_{1+\epsilon}$ for small $t > 0$.

Now to patch the two 1-forms α_t and β_t together we consider the region $A = \overline{N \backslash N_{1+\epsilon}}$. We use the above coordinates on N as coordinates on A. Near the boundary of $M \backslash N$ in A the contact 1-form is $\beta t = dz + t(1+\epsilon+s)d\theta$. We use the map $\Psi(r, \theta, \phi) = (r - 1 - \epsilon, -\phi, \theta)$ to map $A \subset N$ to a neighborhood of the boundary of $M \backslash N_{1+\epsilon}$. This map is orientation preserving and when N is glued to $M \backslash N_{1+\epsilon}$ using this map we recover M. Pulling β_t back to A using this map we get $\Psi^* \beta_t = -trd\phi + d\theta$. We think of this form as defined only near $T_{1+2\epsilon} = \partial N_{1+2\epsilon}$ in A. Similarly $\alpha_t = (1+tr^2)\,d\theta + t(1+f(r))d\phi$ is a form defined near $T_{1+\epsilon} = \partial N_{1+\epsilon}$ in A.

In order to interpolate between these two forms we consider forms on A of the type $\gamma = g(r)d\phi + h(r)d\theta$. This will be a contact form if and only if $g(r)h\prime(r)$-$h(r)g\prime(r) \neq 0$. If we take $g(r)$ and $h(r)$ to be defined by $\Psi^*\beta_t$ and α_t near the boundary of A, then we can clearly extend $g(r)$ and $h(r)$ to all of A so that we have a contact form on A. Moreover, it is easy to check that we can choose $g(r)$ so that $g\prime(r) < 0$ in A.

Let α_t be the 1-from on M that equals α_t on $N_{1+\epsilon}$, β_t on $M\backslash N$ and the form $g(r)d\phi + h(r)d\theta$ on A. This gives a well defined form for all $t \geq 0$. Moreover, α_0 is the form α above that defines the foliation ζ and for small $t > 0$, α_t defines a contact structure $\xi_t = ker\alpha_t$. Thus the contact structure ξ_t is clearly a deformation of the foliation ζ.

We are left to show that ξ_t is supported by the Open Book (L, π). For this we need to check that,

- $\alpha_t(L) > 0$ and (ii) $d\alpha_t|_{page}$ is a volume form on Σ.

- A component of L corresponds to $r = 0$ in N and its positively oriented tangent vector is given by $\frac{\partial}{\partial\phi}$. So $\alpha_t\left(\frac{\partial}{\partial\phi}\right) = d\phi\left(\frac{\partial}{\partial\phi}\right) = 1 > 0$

To check the second condition we consider the four regions $N_1, N_{1+\epsilon}\backslash N_1, A$ and $M\backslash N$. On N_1 the pages of the Open Book correspond to constant θ annuli. The form $d\alpha_t$ restricted to this annulus is $(tf'(r) - \lambda'(r))\, dr \wedge d\phi$ which is never zero and the coefficient is always negative in N_1, but the orientation on the annulus that allows for L to be properly oriented corresponds to the form $d\phi \wedge dr$. So $d\alpha_t$ is a properly oriented non-zero 2-form on the pages in N_1. Now on $N_{1+\epsilon}\backslash N_1$ the pages are still constant θ annuli and the 1-form restricts to $tf\prime(r)dr \wedge d\phi$ on these. Thus α_t is compatible withe the pages in this region. On A the pages are again constant θ annuli, so the form restricted to this is $g\prime(r)dr \wedge d\phi$. By the choice of g this is a properly oriented non-zero 2-form on the pages in A. Finally in $M\backslash N$ $d\beta_t$ is a properly oriented non-zero 2-form on the pages by construction. This completes the proof of our theorem.

5.4 Changing the foliation

Now we examine some related constructions. We can choose other foliations naturally related to the Open Book, and we can study what kinds of contact structures these deform to. Looking at the construction above closely, we realize that there are certain choices we can make while constructing the foliation, which leads to the following questions, What happens when we spin the pages the other way or change the direction of the Reeb components?

We fix a direction for the Reeb component and consider the various cases. Suppose the outward normal (coming out of the page) gives us a positive Reeb component, i.e., the positive normal to the foliation plane at a point on the binding matches with the positive orientation of the binding. The cases to consider then are,

1. Positive Reeb with pages spiralling clockwise.
2. Positive Reeb with pages spiralling anti-clockwise.
3. Negative Reeb with pages spiralling clockwise.
4. Negative Reeb with pages spiralling anti-clockwise.

The construction used in the proof corresponds to the Case 1. Based on the discussion in [11], if we spiral the other way or change the direction of the Reeb component we expect to get an over-twisted contact structure when we perturb our foliation, regardless of the contact structure we start with. This means if we start with a tight contact structure, we perturb to a contact structure not supported by the Open Book.

5.4.1 Constructing the foliations

To understand the above we begin by looking at the foliation in case 1 and try constructing the foliations in the other cases.

Recall from the proof of Theorem 5.1.1. The 1-form defining the foliation is given by

$$\alpha = \begin{cases} \lambda(r)dr + (1-\lambda(r))d\phi & \text{for } r \leq 1 \\ \delta(r)d\theta + (1-\delta(r))dr & \text{for } r > 1 \end{cases}$$

where $\lambda(r)$ equals zero on $[0, \frac{1}{3}]$, equals one for $r \geq 1$ and is strictly increasing on $[\frac{1}{3}, 1]$,

and $\delta(r)$ equals zero on $[0, 1]$, equals one for $r \geq 1 + \epsilon$ and is strictly increasing on $[1, 1 + \epsilon]$.

The coordinates on the Open Book are (r, θ, ϕ) where $r = 0$ on the binding and increases as we move out, θ increases clockwise and the pages correspond to fixed θ and ϕ is the coordinate along the binding.

The positive area form on the pages is given by $d\phi \wedge dr$.

For $r \in [0, \frac{1}{3}]$ the form is given by,

$$\alpha_t = d\phi$$

So the foliation for $0 \leq r \leq \frac{1}{3}$ are given by the constant ϕ plane.

$$\alpha = \begin{cases} dr & \text{for } r = 1 \\ d\theta & \text{for } r > 1 + \epsilon \end{cases}$$

So if we look at the foliation plane near the boundary of the Reeb component the positive normal to the pages point outwards.

Since everything is rotationally symmetric, we can roughly sketch how the plane fields look along a non-compact leaf when viewed along the binding. For this we observe that the normal to the planes points in the ϕ direction at the centre and slowly shifts to the radially outward direction as we move out towards the boundary.

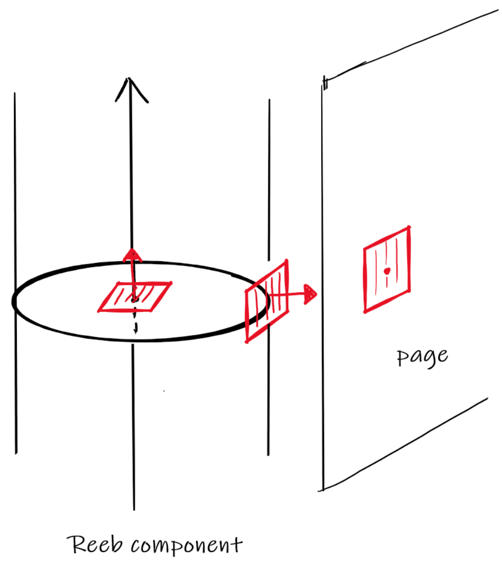


Figure 5.1: Positive Reeb component

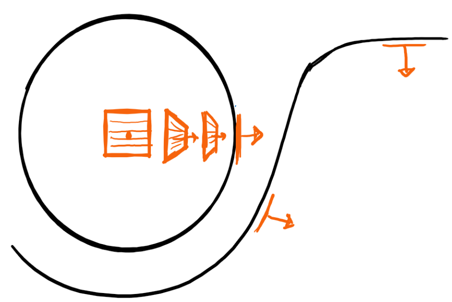


Figure 5.2: Positive Reeb foliation with clockwise spiralling pages (top view: dot denotes arrow pointing out of the page)

Negative Reeb

Next we look at the case of inverting the Reeb component. This looks like we have inverted the direction the inner leaves of the foliation. The foliation plane along the binding point in the opposite direction compared to case 1. The pages are left unchanged. This can be written in terms of a form as follows,

$$\alpha_{-r,s} = \begin{cases} \lambda(r)dr - (1-\lambda(r))d\phi & \text{for } r \leq 1 \\ \delta(r)d\theta + (1-\delta(r))dr & \text{for } r > 1 \end{cases}$$

We can check that this gives us a foliation.

Note that this formula differs from the previous one with respect to the sign of the $d\phi$ component. This seems to be the natural candidate as we have reversed the direction of our foliation plane whose normal point along the ϕ direction.

Note the direction of the normal to plane at $r = 1$ is still the same, so it matches up smoothly with the direction of normal on the foliation planes along pages.

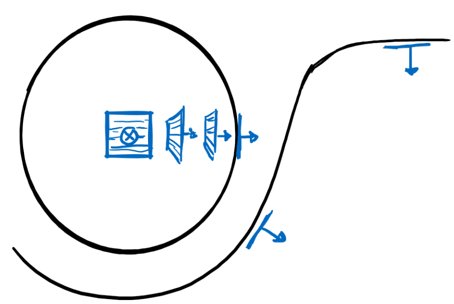


Figure 5.3: Negative Reeb foliation with clockwise spiralling pages (top view: cross denotes arrow pointing away from page)

Spiralling anticlockwise

Now we consider the case where we leave the Reeb component unchanged but spiral the pages the other way. The form defining the foliation is given by,

$$\alpha_{r,-s} = \begin{cases} \lambda(r)dr + (1-\lambda(r))d\phi & \text{for } r \leq 1 \\ -\delta(r)d\theta + (1-\delta(r))dr & \text{for } r > 1 \end{cases}$$

Note : In this case the direction of the normal on the planes along the pages is reversed which makes sure that the planes match up with those along the boundary torus.

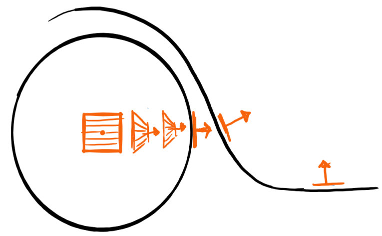


Figure 5.4: Positive Reeb foliation with anticlockwise spiralling pages (top view: dot denotes arrow pointing out of the page)

Negative Reeb with anticlockwise spiralling

$$\alpha_{-r,-s} = \begin{cases} \lambda(r)dr - (1-\lambda(r))d\phi & \text{for } r \leq 1 \\ -\delta(r)d\theta + (1-\delta(r))dr & \text{for } r > 1 \end{cases}$$

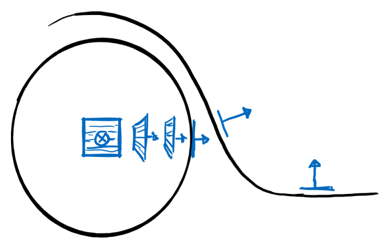


Figure 5.5: Negative Reeb foliation with anticlockwise spiralling pages (top view: cross denotes arrow pointing away from page)

5.4.2 Perturbing the foliations to contact structures

We consider the Negative Reeb case (Case 3) in detail to understand the perturbation in to a contact structure.

We begin by considering the same perturbation as in the case of positive Reeb (Case 1).

$$\alpha_{-r,s_t} = \alpha_{-r,s} + t(r^2 d\theta + (1 + f(r))d\phi)$$

$$\alpha_{-r,s_t} = \begin{cases} \lambda(r)dr - (1-\lambda(r))d\phi + tr^2 d\theta + t(1+f(r))d\phi & \text{for } r \leq 1 \\ \delta(r)d\theta + (1-\delta(r))dr + tr^2 d\theta + t(1+f(r))d\phi & \text{for } r > 1 \end{cases}$$

$$d\alpha_{-r,s_t} = \begin{cases} \lambda'(r)dr \wedge d\phi + 2trdr \wedge d\theta + tf'(r)dr \wedge d\phi & \text{for } r \leq 1 \\ \delta'(r)dr \wedge d\theta + 2trdr \wedge d\theta + tf'(r)dr \wedge d\phi & \text{for } r > 1 \end{cases}$$

Thus we have,

$$\alpha_{-r,s_t} \wedge d\alpha_{-r,s_t} =$$

$$\begin{cases} (-2tr(1-\lambda(r)) - \lambda'(r)tr^2 - t^2r^2f'(r) + 2t^2r(1+f(r)))d\phi \wedge d\theta \wedge dr & \text{for } r \leq 1 \\ (-\delta(r)tf'(r) - t^2r^2f'(r) + t\delta'(r)(1+f(r)) + 2t^2r(1+f(r)))d\phi \wedge d\theta \wedge dr & \text{for } r > 1 \end{cases}$$

Clearly for $r > 1$ this is a positive contact structure for all t since each of the four terms are positive. But for $r \leq 1$ we have two negative and two positive terms therefore it may not be a contact structure for all values of t. But we are interested in finding a perturbation of our foliation to a contact structure through contact structures, so we can then ask if we are contact in a neighbourhood of $t = 0$.

For the above to be a positive contact structure on the interior of the Reeb component it must satisfy,

$$2tr(1-\lambda(r)) + \lambda'(r)tr^2 < -t^2r^2f'(r) + 2t^2r(1+f(r))$$

i.e.,

$$\frac{2(1-\lambda(r)) + \lambda'(r)r}{-rf'(r) + 2(1+f(r))} < t$$

The term on the left hand side is always positive. Thus the perturbation does not result in a positive contact structure in a neighbourhood of 0, in fact it gives us a negative contact structure.

We have a positive contact structure on the outside but we see that we start out as a negative contact structure on the inside. So depending on the choices of our functions there is some place we will have to transition from negative and positive contact structures which cannot be done continuously. Hence, the same perturbation does not give us a contact structure for the negative Reeb case. This leads us to look for other possible perturbations of our foliation.

We consider the following natural choices of perturbations,

$$\alpha_{-r,s_t} = \alpha_{-r,s} - t(r^2 d\theta + (1+f(r))d\phi)$$
$$\alpha_{-r,s_t} = \alpha_{-r,s} + t(r^2 d\theta - (1+f(r))d\phi)$$
$$\alpha_{-r,s_t} = \alpha_{-r,s} - t(r^2 d\theta - (1+f(r))d\phi)$$

We will show that in the case of $\alpha_{-r,s_t} = \alpha_{-r,s} - t(r^2 d\theta + (1+f(r))d\phi)$, we do not perturb to a contact structure.

$$\alpha_{-r,s_t} = \begin{cases} \lambda(r)dr - (1-\lambda(r))d\phi - tr^2 d\theta - t(1+f(r))d\phi & \text{for } r \leq 1 \\ \delta(r)d\theta + (1-\delta(r))dr - tr^2 d\theta - t(1+f(r))d\phi & \text{for } r > 1 \end{cases}$$

$$d\alpha_{-r,s_t} = \begin{cases} \lambda'(r)dr \wedge d\phi - 2tr dr \wedge d\theta - tf'(r)dr \wedge d\phi & \text{for } r \leq 1 \\ \delta'(r)dr \wedge d\theta - 2tr dr \wedge d\theta - tf'(r)dr \wedge d\phi & \text{for } r > 1 \end{cases}$$

Thus we have,

$\alpha_{-r,s_t} \wedge d\alpha_{-r,s_t} =$

$$\begin{cases} (2tr(1-\lambda(r)) + \lambda'(r)tr^2 - t^2r^2 f'(r) + 2t^2 r(1+f(r)))d\phi \wedge d\theta \wedge dr & \text{for } r \leq 1 \\ (\delta(r)tf'(r) - t^2r^2 f'(r) - t\delta'(r)(1+f(r)) + 2t^2 r(1+f(r)))d\phi \wedge d\theta \wedge dr & \text{for } r > 1 \end{cases}$$

The contact structure is positive on the Reeb component for all values of t,as all the terms are positive (Recall the term wise analysis from the original proof). For the contact structure to be positive outside the Reeb component,

$$t > \frac{-(\delta f'(r) - \delta'(r)(1+f(r)))}{r(2(1+f(r)) - rf'(r))}$$

This implies it is not a positive contact structure in an open neighbourhood of $t = 0$, but for the same t value it is positive on the interior of the Reeb component. Thus we cannot deform continuously deform in to a contact structure under this perturbation.

Next we show that in the case of $\alpha_{-r,s_t} = \alpha_{-r,s} + t(r^2 d\theta - (1+f(r))d\phi)$ we perturb to a negative contact structure,

$$\alpha_{-r,s_t} = \begin{cases} \lambda(r)dr - (1-\lambda(r))d\phi + tr^2 d\theta - t(1+f(r))d\phi & \text{for } r \leq 1 \\ \delta(r)d\theta + (1-\delta(r))dr + tr^2 d\theta - t(1+f(r))d\phi & \text{for } r > 1 \end{cases}$$

$$d\alpha_{-r,s_t} = \begin{cases} \lambda'(r)dr \wedge d\phi + 2trdr \wedge d\theta - tf'(r)dr \wedge d\phi & \text{for } r \leq 1 \\ \delta'(r)dr \wedge d\theta + 2trdr \wedge d\theta - tf'(r)dr \wedge d\phi & \text{for } r > 1 \end{cases}$$

Thus we have,

$$\alpha_{-r,s_t} \wedge d\alpha_{-r,s_t} =$$

$$\begin{cases} (-2tr(1-\lambda(r)) - \lambda'(r)tr^2 + t^2r^2f'(r) - 2t^2r(1+f(r)))d\phi \wedge d\theta \wedge dr & \text{for } r \leq 1 \\ (\delta(r)tf'(r) + t^2r^2f'(r) - t\delta'(r)(1+f(r)) - 2t^2r(1+f(r)))d\phi \wedge d\theta \wedge dr & \text{for } r > 1 \end{cases}$$

By considering the constraints on the function in each term we see that the above is negative for all values of r and t.

Thus the foliation perturbs to a negative contact structure for all values of t and r.

Finally in the case of $\alpha_{-r,s_t} = \alpha_{-r,s} - t(r^2 d\theta - (1+f(r))d\phi)$ we show that we perturb to a positive contact structure in a neighbourhood of 0,

$$\alpha_{-r,s_t} = \begin{cases} \lambda(r)dr - (1-\lambda(r))d\phi - tr^2 d\theta + t(1+f(r))d\phi & \text{for } r \leq 1 \\ \delta(r)d\theta + (1-\delta(r))dr - tr^2 d\theta + t(1+f(r))d\phi & \text{for } r > 1 \end{cases}$$

$$d\alpha_{-r,s_t} = \begin{cases} \lambda'(r)dr \wedge d\phi - 2trdr \wedge d\theta + tf'(r)dr \wedge d\phi & \text{for } r \leq 1 \\ \delta'(r)dr \wedge d\theta - 2trdr \wedge d\theta + tf'(r)dr \wedge d\phi & \text{for } r > 1 \end{cases}$$

Thus we have,

$\alpha_{-r,s_t} \wedge d\alpha_{-r,s_t} =$

$$\begin{cases} (2tr(1-\lambda(r)) + \lambda'(r)tr^2 + t^2r^2f'(r) - 2t^2r(1+f(r)))d\phi \wedge d\theta \wedge dr & \text{for } r \leq 1 \\ (-\delta(r)tf'(r) + t^2r^2f'(r) + t\delta'(r)(1+f(r)) - 2t^2r(1+f(r)))d\phi \wedge d\theta \wedge dr & \text{for } r > 1 \end{cases}$$

For $\alpha_{-r,s_t} \wedge d\alpha_{-r,s_t} > 0$ we require,

$$2tr(1-\lambda(r)) + \lambda' tr^2 > 2t^2r(1+f(r)) - t^2r^2f'(r)$$

and

$$-t\delta(r)f'(r) + t\delta'(r)(1+f(r)) > 2t^2r(1+f(r)) - t^2r^2f'(r)$$

$$\begin{array}{ll} t < \frac{2(1-\lambda(r))+\lambda'(r)r^2}{2r(1+f(r))-r^2f(r)} & \text{for } r \leq 1 \\ t < \frac{-\delta(r)f'(r)+\delta'(r)(1+f(r))}{2r(1+f(r))-r^2f(r)} & \text{for } r > 1 \end{array}$$

Thus for t values less than the minimum of the above two right hand terms we perturb to a positive contact structure, i.e., we have a positive contact structure in a neighbourhood of $t = 0$.

So now we consider the contact structure at a t value in this range.

We want to show the perturbed contact structure is overtwisted in case of negative Reeb (or anticlockwise spiralling).

5.4.3 Overtwisted or Tight?

We will show that the case $\alpha_{-r,s_t} = \alpha_{-r,s} - t(r^2d\theta - (1+f(r))d\phi)$ is overtwisted by explicitly finding an overtwisted disk. By rotational symmetry along θ and ϕ we can reduce this to a 1-dimensional problem.

We consider a meridional disc of radius r. A vector field along the boundary of of such a disc is given by $\frac{\partial}{\partial\theta}$.

Since

$$\alpha_{-r,s_t} = \begin{cases} \lambda(r)dr - (1-\lambda(r))d\phi - tr^2 d\theta + t(1+f(r))d\phi & \text{for } r \leq 1 \\ \delta(r)d\theta + (1-\delta(r))dr - tr^2 d\theta + t(1+f(r))d\phi & \text{for } r > 1 \end{cases}$$

$$\alpha_{-r,s_t}(\frac{\partial}{\partial\theta}) = \begin{cases} -tr^2 & \text{for } r \leq 1 \\ \delta(r) - tr^2 & \text{for } r > 1 \end{cases}$$

This tells us that for discs with radius less than 1 the boundary is a Transverse curve as $-tr^2$ is non-zero.

For a disc with radius r_0 greater than 1, the condition for the boundary to be a transverse curve is $\delta(r_0) - tr_0^2 < 0$ since $\alpha_{-r,s_t}(\frac{\partial}{\partial\theta})$ is continuous i.e.,

$$t > \frac{\delta(r_0)}{r_0^2}$$

So if we choose $t < min\{\frac{\delta(r_0)}{r_0^2}, \frac{-\delta(r_0)f'(r_0)+\delta'(r_0)(1+f(r_0))}{2r_0(1+f(r_0))-r_0^2 f(r_0)}\}$ we have a positively Transverse curve. This means we have passed through a Legendrian curve.

We choose the minimum such r value. The disc with this radius gives us a disc with Legendrian boundary. Now we look at the characteristic foliation of this disc. We observe an elliptic singularity at the centre. Next we look for other singularities. By rotational symmetry, any other singularity would result in a Legendrian curve, which is a contracdiction to our choice of minimum r value. Therefore this is an overtwisted disc.

This tells us the contact structure is overtwisted. This implies that if to begin with we choose a tight contact structure we perturb to a contact structure not supported by the Open Book. Thus the choice of direction made in the proof is crucial for the proof.

Also since tight foliations perturb to tight contact structures[10], for an overtwisted contact structure any foliation which perturbs to it must have a Reeb component, since Reebless foliations are tight.

Chapter 6

Conclusions and Future Scope

6.1 Conclusions

In this study we looked at various aspects of contact structures in a 3-manifold setting, specifically their relations to codimension-1 foliations. We began by discussing contact structures in detail; definitions, existence, classifications, etc. In particular, we studied different topological objects, namely knots, braids and foliations related to contact manifolds. Next we explored Open Book decomposition of manifolds, their constructions, and relations pertaining to contact manifolds. Open Books serve as a bridge between topological and differential aspects of contact topology, by specifying equivalence classes of contact manifolds. Following this, as an example to elucidate the role of Open Books in the study of contact manifolds, we investigated plane field structures in greater generality and looked at relations of contact manifolds to foliations in the same homotopy class. Following the result of Eliashberg and Thurston that every foliation can be perturbed to a contact structure [17], we looked at Etnyre's proof of the converse, i.e., every contact structure is close to a foliation [11]. The proof relies on Open Books, in particular Giroux correspondence [14].

In Etnyre's proof, for a given contact structure a foliation was constructed which is compatible with an Open Book supporting the contact structure. Construction of foliation involves replacing the binding with a Reeb component and spiralling the pages towards it in a clockwise manner. The proof leads us to the explore various related foliations and contact structures associated to them. We obtained the following results from our study of these foliations,

- We constructed foliations by changing the direction of the Reeb component and direction of spiralling.
- When we change the direction of one of the above, we found a perturbation under which the foliation perturbs to a positive contact structure. We also showed that for other related perturbations we can perturb to a negative contact structure.
- In case of the positive contact structure we proved that the perturbed contact structure is overtwisted.
- Given an initial choice of tight contact structure and its associated foliation, we were able to perturb the foliation to a constact structure not supported by the Open Book.
- We also looked at generalizations to confoliations.

6.2 Future scope

A natural next step would be to explore further relations between contact structures and foliations, in particular relations between tight contact structures and taut/reebless foliation.

The foliation we construct has a Reeb component. So the question that arises then is that can be find a tight foliation close to a tight contact structure. Or more specifically is every tight contact structure a deformation of a tight (Reebless) foliation?

Appendix A

Braid and knots in a contact 3-manifold

We look at another example of an application of Open Books in the study of contact topology.

Alexander's theorem tells us that any knot in $\mathbb{R}^3$ can be braided about the z-axis. This naturally leads us to defining a more general notion of braids in a contact 3-manifold using an Open Book decomposition of the manifold that supports the contact structure.

Let (L, π) be an Open Book decomposition for M. A link $K \subset M$ is said to be braided about L if K is disjoint from L and there exists a parametrization of K,$f : \coprod S^1 \to M.$, such that if θ is the coordinate on each S^1 then $\frac{d}{d\theta}(\pi \circ f) > 0, \forall \theta$. We call bad arcs of K those arcs where this condition is not satisfied.

We observe that such braids are naturally transverse to the contact structure, which leads to the question: Are all tranverse links braidable. This paper shows that in fact any tranverse link in a contact manifold can be tranversely braided with respect to an Open Book that supports the contact structure.

Theorem A.0.1. *(Bennequin [4]) Any transverse link in* $(\mathbb{R}^3, \xi_{std})$ *is transversely isotopic to a link braided about the z-axis.*

The following theorem is a generalization of this theorem:

Theorem A.0.2. *(Pavlescu[15]) Suppose* (L, π) *is an Open Book decomposition for a 3-*

manifold M and ξ is supported by (L, π). Let K be a transverse link in M. Then K can be transversely isotoped to a braid.

The idea of the proof is to find a family of diffeomorphisms of M that fixes each page of the Open Book setwise and sends the parts of the link where the link is not braided into a neighbourhood of the binding. The neighbourhood of the binding is contactomorphic to a neighbourhood of the z-axis in $(\mathbb{R}^3, \xi_{std})$ this claim is an extension of Darboux's Theorem which states that all contact structures look the same locally around a point. Then the link can be braided using Theorem 3.2.

We can construct such a family diffeomorphism as a flow of a vector field tangent to the pages. The singularities thus occur at the places where the tangent to the page coincides with the contact plane. Thus, to be able to move the bad arcs into the neighbourhood of the binding we need to isolate them from the singular points. This is achieved via the process of "wrinkling" (introducing wrinkles in along K), the process is demonstrated in the following figure.

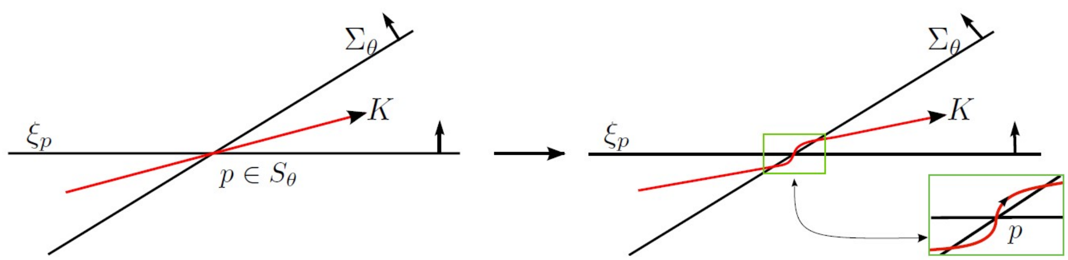


Figure A.1: Wrinkling K to avoid intersection with singularities[15]

The wrinkles may sometimes increase the number of arcs but these new arcs avoid the singular points.

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