Contact structures and foliations

An application of open books in contact topology

Final Presentation

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Contact structure

A contact structure is a nowhere integrable plane field on a manifold.

Definition 1 (Contact Structure)

A (oriented) contact structure ξ on M is an oriented plane $\xi \subset TM$ for which there is a 1-form α such that $\xi = \ker(\alpha)$ and $\alpha \wedge d\alpha > 0$.

Introduction

Why open book?

An open book is a topological decomposition of a 3-manifold that also specifies equivalence classes of contact structures.

Definition 2 (Open Book)

An open book decomposition of M is a pair (B, π) where

1. B is an oriented link in M called the binding of the open book and
2. $\pi : M \backslash B \to S^1$ is a fibration of the complement of B such that $\pi^{-1}(\theta)$ is the interior of a compact surface $\Sigma_\theta \subset M$ and $\partial \Sigma_\theta = B$ for all $\theta \in S^1$. The surface $\Sigma = \Sigma_\theta$ for any θ, is called the page of the open book.

Contact structures and open book

A contact structure ξ on M is said to be **supported** by an open book (B, π) if ξ can be isotoped through contact structures so that there is a contact 1-form α for ξ such that

1. $d\alpha$ is a positive area form on each page Σ_θ of the open book and
2. $\alpha > 0$ on the tangent to the oriented binding B.

Theorem 1

Every open book decomposition supports a contact structure.

Contact structures and open book

Theorem 2 (Giroux)

Let M be a closed oriented 3-manifold. Then there is a one to one correspondence between

{oriented contact structures on M up to isotopy}
and
{open book decompositions of M up to positive stabilization}.

Foliations can be approximated by contact structures on a closed oriented 3-manifold (Eliashberg and Thurston).

Question

Is every contact structure on a 3-manifold "close" to a foliation?

- Close here means being a deformation.
- A contact structure ξ is said to be a deformation of a foliation ζ if there is a one parameter family of plane fields ξ_t such that $\xi_0 = \zeta$ and $\xi_1 = \xi$, and ξ_t is a contact structure for $t > 0$.

Theorem 3 (Etnyre)

Every positive and negative contact structure on a closed oriented 3-manifold is a C^∞-deformations of a C^∞-foliations.

Outline of Proof:

- We start with a contact structure ξ
- Then we choose some open book (B, π) that supports ξ
- We construct a foliation on M associated with the chosen open book.
- Next we show that we can perturb the foliation into a contact structure supported by our open book
- Finally by Giroux correspondence we can conclude that the perturbed contact structure is isotopic to ξ since they are supported by the same open book. Thus proving our theorem.

Contact structures and foliations
Constructing the foliation

We replace the neighbourhood of the binding with Reeb components and spin the pages of the open book to limit to the Reeb components.

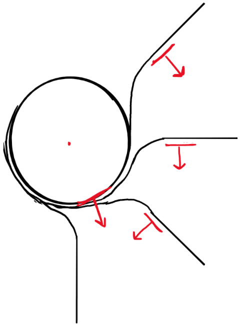

Figure: View along the Binding

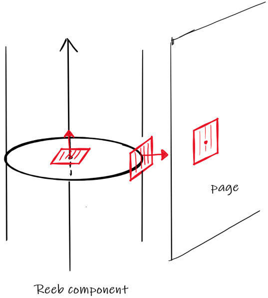

Figure: Reeb component with a Page

Other related foliations

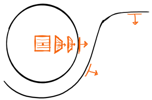

Figure: Positive Reeb with Clockwise spiralling

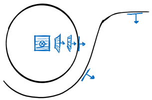

Figure: Negative Reeb with Clockwise spiralling

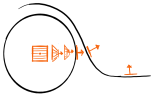

Figure: Positive Reeb with Anticlockwise spiralling

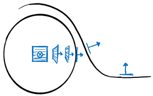

Figure: Negative Reeb with Anticlockwise spiralling

Conclusions

Main results:

- Describing related foliations in terms of 1-form.
- Pertubation to a positive contact structure in each of the above foliation cases.
- Upon reversing the direction of Reeb component or spiralling of pages (not both) the contact structure is overtwisted.

Definition 3 (Confoliation)

A plane field $\xi = \ker \alpha$ on an oriented manifold is called a positive (negative) confoliation if $\alpha \wedge d\alpha \geq 0$ ($\alpha \wedge d\alpha \leq 0$).

- If ξ is a tight foliation we can isotop ξ through tight confoliations to a foliation with Reeb components.
- A taut foliation is isotopic to a Reeb foliation via tight confoliations.
- If a contact structure is virtually overtwisted then it must be a perturbation of a Reeb foliation. Since tight foliations perturb to tight contact structures.

References

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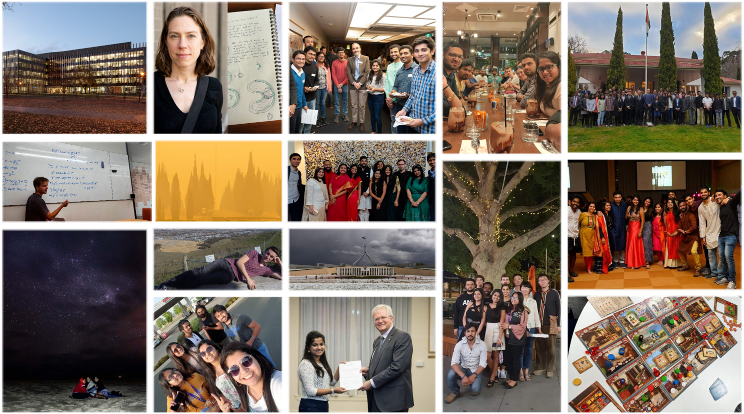
Memories from the other hemisphere



THANK YOU!